KEY

Math 251
Limits and Derivatives of Trig §3.3

Get into your assigned group of two or three people and work on the following problems. You are welcome to compare your answers with other groups but only after you have completed the problem. Also feel free to use the book and your notes as a reference. One copy from each group should be turned in by Tuesday at 6pm. Make sure that your answers are written up completely and clearly (with correct notation!!!) as there will be no opportunity for rewriting problems that you get wrong.

1. Recall that the definition of derivative gives us:

$$(\cos x)' = \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

Finish the proof that $\frac{d}{dx}(\cos x) = -\sin x$ by completing the following steps:

(a) Find $\lim_{h\to 0} \cos x$ using theorems we have covered in class.

$\lim_{h\to 0} \cos x = \cos x$ since $\cos x$ is independent of h. In the eyes of the limit $\cos x$ is constant

(b) Find $\lim_{h\to 0} \sin x$ using theorems we have covered in class.

$\lim_{h\to 0} \sin x = \sin x$. The limit as h goes to 0 of a function that does <u>not</u> depend on h is just the original function

(c) Find $\lim_{h\to 0} \frac{\sin h}{h}$ using theorems we have covered in class.

$\lim_{h\to 0} \frac{\sin h}{h}$ by work involving the squeeze thm done in class today.

(d) Consider $\lim_{h\to 0} \frac{\cos h-1}{h}$. Multiply this expression by $\frac{\cos h+1}{\cos h+1}$ (looks familiar right?!) and try evaluating the limit. Be sure to cite any identities that were used.

$$\lim_{h\to 0} \frac{\cos h - 1}{h}\left(\frac{\cos h+1}{\cos h+1}\right) = \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h+1)}$$

note: $\sin^2 h + \cos^2 h = 1 \Rightarrow \cos^2 h - 1 = -\sin^2 h$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h+1)} = \lim_{h\to 0} \frac{-\sin h}{h} \frac{\sin h}{(\cos h+1)}$$

$$= -\lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{\sin h}{\cos h+1} = -1 \cdot \frac{0}{2} = 0$$

(e) Put all parts a through d together and write down what $(\cos x)'$ is equal to.

$$(\cos x)' = \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\cos h-1}{h} - \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x$$

Remark: Similar work to this is done in the book to show $(\sin x)' = \cos x$

2. Find $\frac{d}{dx}\csc x$ by following the following steps:

(a) Define $\csc x$ in terms of other trigonometric functions whose derivatives you know. ($\sin x$ and $\cos x$)

$$\csc x = \frac{1}{\sin x}$$

(b) Rewrite $\frac{d}{dx}\csc x$ in terms of these other trigonometric functions you found in the previous question. To take the derivative what derivative rule would you like to use? (Power rule? Product Rule? etc)

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$ Quotient Rule.

(c) Take the derivative of $\frac{d}{dx}\csc x$ and confirm your work with the book or a neighbor group.

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(1) - 1\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x \cdot 0 - \cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \cdot \csc x$$

3. Recall for *ALL* θ we have shown in class that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, even if $\theta = 3x$. Find the following but note we are looking for *limits* NOT derivatives.

(a) Find $\lim_{x\to 0}\frac{\sin 3x}{3x}$.

$$\lim_{x\to 0}\frac{\sin 3x}{3x} = \lim_{\frac{1}{3}\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

(b) Find $\lim_{x\to 0}\frac{\sin 3x}{x}$. (Hint: remember that multiplying by $1 = \frac{3}{3}$ won't change the limit!!).

$$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3}{3}\cdot\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x} = 3\cdot 1 = 3.$$

(c) Find $\lim_{t\to 0}\frac{\tan 6t}{\sin 2t}$. (Hint: remember that multiplying by $1 = \frac{6t}{6t}$ won't change the limit!!!)

$$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\frac{\sin 6t}{\cos 6t}\cdot\frac{1}{\sin 2t} = \lim_{t\to 0}\frac{6t}{6t}\,\frac{\sin 6t}{\cos 6t}\,\frac{1}{\sin 2t}$$

$$= \lim_{t\to 0}\frac{6t}{\cos 6t}\cdot\frac{\sin 6t}{6t}\cdot\frac{1}{\sin 2t}$$

$$= \lim_{t\to 0}\frac{6t}{\cos 6t}\cdot\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\sin 2t}\left(\frac{2t}{2t}\right)$$

$$= \lim_{t\to 0}\frac{6t}{\cos 6t}\cdot 1\cdot\lim_{t\to 0}\frac{1}{\left(\frac{\sin 2t}{2t}\right)}\cdot\frac{1}{2t} = \lim_{t\to 0}\frac{6t}{2t}\,\frac{1}{\cos 6t}$$

$$= 6/2 = 3$$