Ave 77%
Mean 75%

Key

Exam 2 | TMath 126 /40 | Winter 2024

As a reminder, you are welcome to use a two-sided 3.5" by 5" index card with notes (written or typed), a non-internet accessing calculator (which includes Desmos Test Mode) but no books, other notes, or peers.

1. [6] TRUE/FALSE: Write True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, write False and provide a counterexample or brief justification.

   (a) (3DActivity#3) The limit, $\lim_{(x,y)\to(2,1)} \dfrac{x-2}{x^2y-4y}$ is not defined because the limit evaluates to "0/0".

   +.5 False

   $\dfrac{2-2}{2^2(1)-4(1)} = \dfrac{0}{4-4} = \text{"}\dfrac{0}{0}\text{"}$

   +.5 we know nothing... need other methods

   Is defined?

   start +.5
   use def of limit +1
   sense +.5

   Algebra

   $\lim_{(x,y)\to(2,1)} \dfrac{x-2}{y(x^2-4)} = \lim_{(x,y)\to(2,1)} \dfrac{\cancel{x-2}}{y(x+2)\cancel{(x-2)}}$

   $= \lim_{(x,y)\to(2,1)} \dfrac{1}{y(x+2)} = \dfrac{1}{1(2+2)} = \dfrac{1}{4}$

   OR Numerical

y \ x	1.9	1.99	2.001	2.1
.9				
.99				
1.001				

   (b) (WebHW14.6#2) Given $f(x,y) = x^2\ln(y)$ and $\vec{u} = \langle -5, 1\rangle$ we can compute the directional derivative of f in the direction of $\vec{u}$ at point $(4,1)$ as follows:

   +.5 False

   $$D_{\vec{u}}f(4,1) = \nabla f(4,1)\cdot\langle -5,1\rangle = \underbrace{\langle 2(4)\ln(1), 4^2\tfrac{1}{1}\rangle}_{\text{Checks out.}}\cdot\langle -5,1\rangle = 0+16 = 16$$

   start +.5
   def of $D_u f$ +1
   and unit length +1

   $f(x,y) = x^2\ln(y)$

   $f_x(x,y) = 2x\ln(y) \Rightarrow f_x(4,1) = 2\cdot 4\cdot\ln(1) = 0 \Rightarrow \nabla f(4,1) = \langle 0, 4\rangle$

   $f_y(x,y) = x^2\dfrac{1}{y} \qquad f_y(4,1) = 2^2\cdot\dfrac{1}{1} = 4$

   Recall need $\|\vec{u}\| = 1$. Note $\|\vec{u}\| = \sqrt{(-5)^2+1^2} = \sqrt{26} \neq 1$

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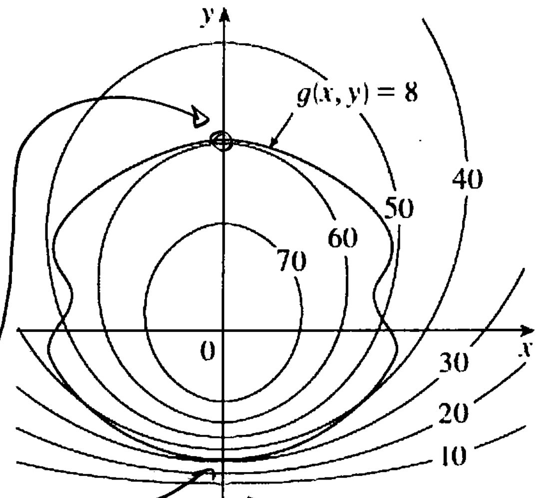
Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [2] (WebHW14.8 #1) A contour map of f & a curve with the equation $g(x, y) = 8$ is shown. Estimate the maximum values of f subject to the constraint that $g(x, y) = 8$.



graph reading (+.5)

(+.5) [happens when contour curves of f are tangent to $g(x,y)=8$.

Value is 60 (+1) (note min is ≈ 30) value close to 60/ on graph +.5

3. [6] For this problem outline (you do not actually need to find!) a solution. Make sure your outline includes:

(a) definitions of variables used,

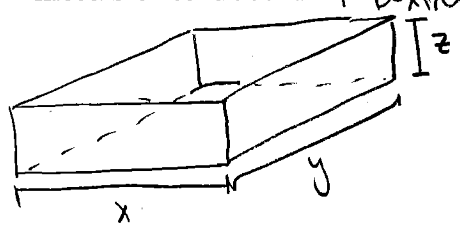
(b) identifying the function that needs to be optimized,

(c) boxing systems of equations that need to be solved (but do not solve them!), &

(d) explaining how you would verify your work is correct (ie a maximum)

(§14.7 ex6 & §14.8 ex2) A rectangular box without a lid is to be made from 12 square meters of cardboard. Maximize the volume.

(+1) a)



b) maximize Volume $= x \cdot y \cdot z$

(1.5) c) subject to constraint

$C = xy + 2xz + 2yz = 12m^2$

Could use Lagrange multipliers find x, y, z, λ ∋

(+.5) $\left\{ \begin{array}{l} (\nabla Vol) = \lambda \langle C_x, C_y, C_z \rangle \\ xy + 2xz + 2yz = 12 \end{array} \right\}$

derivatives (+2)

$$\boxed{\begin{array}{l} yz = \lambda(y+2z) \\ xz = \lambda(x+2z) \\ xy = \lambda(2x+2y) \\ xy+2xz+2yz = 12 \end{array}}$$

OR

b) Maximize Volume $= x \cdot y \cdot z$

(+1.5) Since $xy + 2xz + 2yz = 12$

$\Rightarrow 2xz + 2yz = 12 - xy$

$\Rightarrow z(2x+2y) = 12 - xy$

$\Rightarrow z = \dfrac{12-xy}{2x+2y}$ sub into Max Vol

$Vol(x,y) = x \cdot y\left(\dfrac{12-xy}{2x+2y}\right) = \dfrac{12xy - x^2y^2}{2x+2y}$

c) (1.5) $Vol_x(x,y) = 0$, $Vol_y(x,y) = 0$

(+2)

$$\boxed{\begin{array}{l} 0 = \dfrac{(2x+2y)(12y - 2xy^2) - (12xy - x^2y^2)2}{(2x+2y)^2} \\ 0 = \dfrac{(2x+2y)(12x - 2x^2y) - (12xy - x^2y^2) \cdot 2}{(2x+2y)^2} \end{array}}$$

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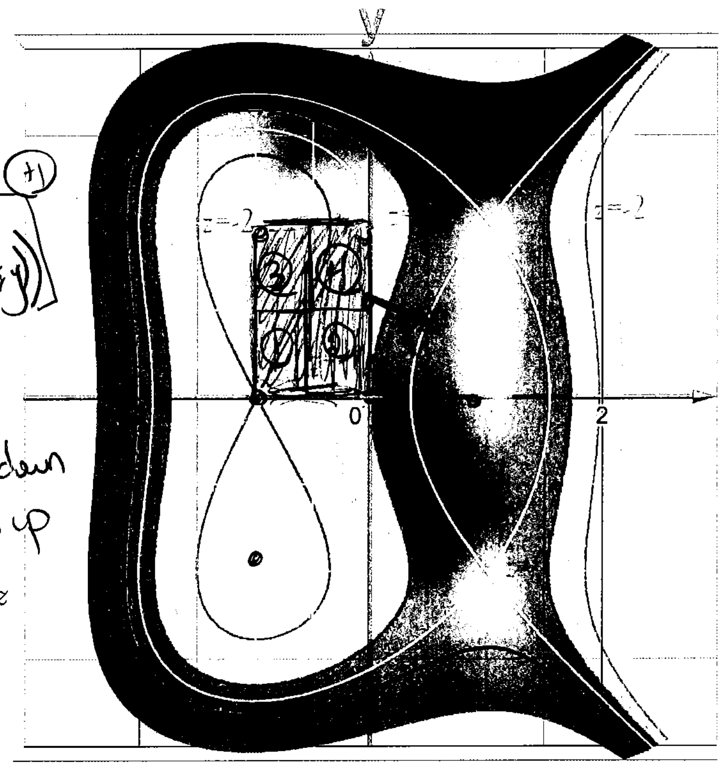
(+1) d) I'd need to check points close by the CP that satisfy constraint & look at Volumes

4. Consider $f(x, y)$ whose contour map is shown on the right.



(a) [2] (WrittenHW14.2#46) Is $f(x, y)$ a function? Explain your reasoning. (+1)

sense (+.5)

(+.5) Yup - Passes the vertical line test (one z value for each (x,y))

(b) [2] (OptimizingActivity#1) Identify (+.5) (-1, 0) as a local minimum, maximum, or saddle. (+1) Saddle

graphreading (+.5)

when move // to y axis goes down
when move // to x axis goes up

(c) [2] (WebHW14.7#2) Use the contour map to predict two extreme points for z (either local minimums or maximums).

graphreading (+.5)
got one (+1)
got two (+.5)

min @ (-1,1) and (-1,-1)
Max @ (1,0)

(d) [2] (Quiz5#1) Determine if $f_x(0, 0)$ is positive, zero, or negative. Explain your reasoning.

(+.5) → in direction // to x axis →

(+.5) @ (0,0)

(+.5) goes up from 0 to 1

(+.5) So $\frac{1}{1} = \frac{\Delta z}{\Delta x} \approx f_x(0,0) \Rightarrow f_x(0,0)$ is positive

(e) [3] (Quiz6#1) *Sketch* the direction of the gradient vector $\nabla f(0, \frac{1}{2})$

@ (0, ½) (+.5)
vector (+.5)

(+1) direction of steepest accent

got it (+1)

(f) [3] (IntegratingActivity#1)) Estimate the signed volume trapped by $f(x, y)$, the xy plane, and above the rectangle bounded by $-1 \le x \le 0$ and $0 \le y \le 1$. Be clear with your choices so I can follow your work! $\int_0^1\int_{-1}^0 f(x,y)\,dx\,dy$ +.5

(+.5) Need to estimate the signed volume trapped above/below the shaded rectangle above, I'll use divide the region

(+.5) into 4 squares $\Delta x = \frac{1}{2}$, $\Delta y = \frac{1}{2}$ (For height, I'll use the lower left corners. (+.5))

Vol of prism (+.5)

prism1 + prism2 + prism3 + prism4 =

$\Delta x \Delta y \cdot \text{height}1 + \Delta x \Delta y \cdot \text{height}2 + \Delta x \Delta y \cdot \text{height}3 + \Delta x \Delta y \cdot \text{height}4$

$\frac{1}{2}\cdot\frac{1}{2}(-2) + \frac{1}{2}\cdot\frac{1}{2}\left(\frac{-3}{2}\right) + \frac{1}{2}\cdot\frac{1}{2}(-3) + \frac{1}{2}\cdot\frac{1}{2}(-2)$

graphread (+1)

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5. A function $f(x,y)$ of two variables is known to be continuous and has the values specified to the right.

$y \setminus x$	1.0	1.1	1.2
2.0	5	7	10
2.2	~~4~~	~~6~~	~~8~~
2.4	3	5	6

(a) [1] What is $f(1.1, 2.4)$? $= 5$

(b) [4] (PracticeExam2#3) Your boss would like you to develop a linear model that could be used to estimate the value of $f(1.4, 2.3)$. Build the model and justify the choices/steps that you make.

start (+.5) We will use a linearization $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$ (+.5)

(+1) Center @ $f(1.2, 2.2)$ which is closer to $x = 1.4$ and $y = 2.3$

$\Rightarrow x_0 = 1.2,\ y_0 = 2.2 \quad z = 8$

(+.5) $m_x \approx \dfrac{\Delta z}{\Delta x} = \dfrac{8-6}{1.2-1.1} = \dfrac{2}{.1} = 20$

(+.5) $m_y \approx \dfrac{\Delta z}{\Delta y} = \dfrac{8-6}{2.2-2.4} = \dfrac{2}{-.2} = -10$

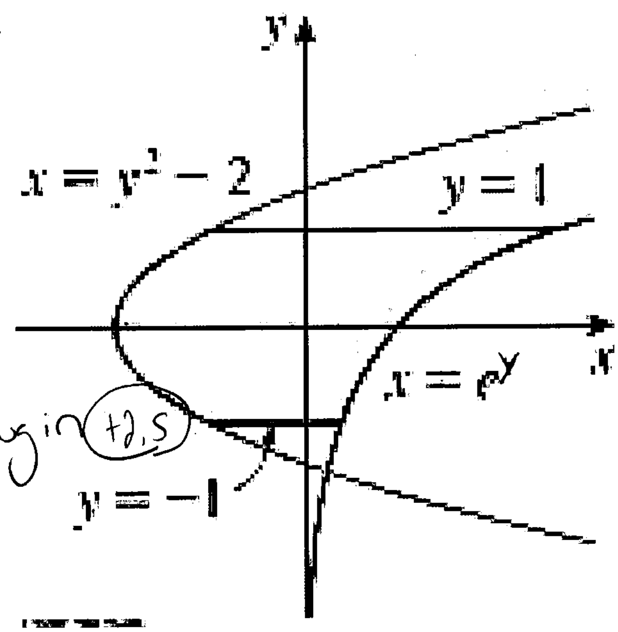
Model:

$\therefore\ z - 8 = 20(x - 1.2) - 10(y - 2.2)$

plug in (+1)

6. (WebHW15.2#5) Consider the volume trapped above the region shaded on the right.

(a) [3] (WebHW14.4#3) The height function, h, is unknown. Use the fact that we know $h(-1,0) = 1$, $h_x(-1,0) \approx \frac{1}{2}$, and $h_y(-1,0) \approx 1$ ~~to find a linear~~ approximation for h.



(+.5) $z - z_0 = m_x(x - x_0) + m_y(y - y_0)$

$z - 1 = \frac{1}{2}(x - -1) + 1(y - 0)$ plug in (+2.5)

$\Rightarrow z = \frac{1}{2}(x+1) + y + 1$

(b) [4] Find an iterated integral of our linear approximation to estimate the volume. (That is, write down the expression so that technology can finish the computations.)

start (+.5)
limits of y (+1)
limits of x (+1)
function integrating (+.5)
notation (+1)

Easiest to fix y + integrate with respect to x first OR

$$\int_{-1}^{1} \int_{\text{Blue}}^{\text{Red}} \tfrac{1}{2}(x+1) + y + 1 \, dx\, dy$$

$$\int_{-1}^{1} \int_{y^2-2}^{e^y} \tfrac{1}{2}(x+1) + y + 1 \, dx\, dy$$

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