Key

Exam 2

TMath 126

Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [12] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) If $\vec{u}\cdot\vec{v} = 0$, then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

   false Let $\vec{u} = \langle 1,0,0\rangle$ and $\vec{v} = \langle 0,1,0\rangle$. Then $\vec{u}\cdot\vec{v} = 0$ but neither $\vec{u}$ nor $\vec{v}$ is zero

   (b) $(\vec{u}\times\vec{w})\cdot\vec{w} = 0$

   True. The vector $\vec{u}\times\vec{w}$ is $\perp$ to both $\vec{u}$ and $\vec{w}$. Thus $(\vec{u}\times\vec{w})\cdot\vec{w} = \|(\vec{u}\times\vec{w})\|\|\vec{w}\|\cos\frac{\pi}{2} = 0$

   (c) $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{\vec{u}}{\|\vec{u}\|}\cdot\dfrac{\vec{v}}{\|\vec{v}\|}$.

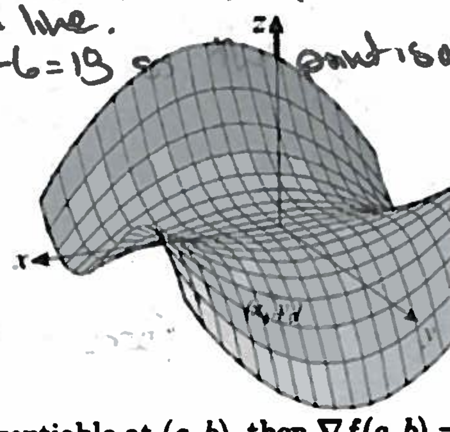
   True Let $\vec{u} = \langle u_1,u_2,u_3\rangle$ and $\vec{v} = \langle v_1,v_2,v_3\rangle$ by properties of mult in $\mathbb{R}$.

   Then $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{u_1v_1+u_2v_2+u_3v_3}{\sqrt{u_1^2+u_2^2+u_3^2}\cdot\sqrt{v_1^2+v_2^2+v_3^2}} = \dfrac{1}{\sqrt{u_1^2+u_2^2+u_3^2}}\langle u_1,u_2,u_3\rangle\cdot\dfrac{1}{\sqrt{v_1^2+v_2^2+v_3^2}}\langle v_1,v_2,v_3\rangle$

   (d) The line $(2+3t, -4t, 5+t)$ where $t\in\mathbb{R}$ intersects the plane $4x+5y-2z = 18$ at the point $(-4,8,3)$.

   True Notice if $t=-2$ then $(2+3(-2), -4(-2), 5+-2) = (-4,8,3)$ ⇒ the point is on the given line.
   Also $4(-4)+5(8)-2(3) = -16+40-6 = 18$ so point is on the plane too

   (e) Consider the function g pictured to the right.
   $g_x(x_0,y_0) > 0$.



   True, as we follow the contour line // to the x-axis, the z coord increases.

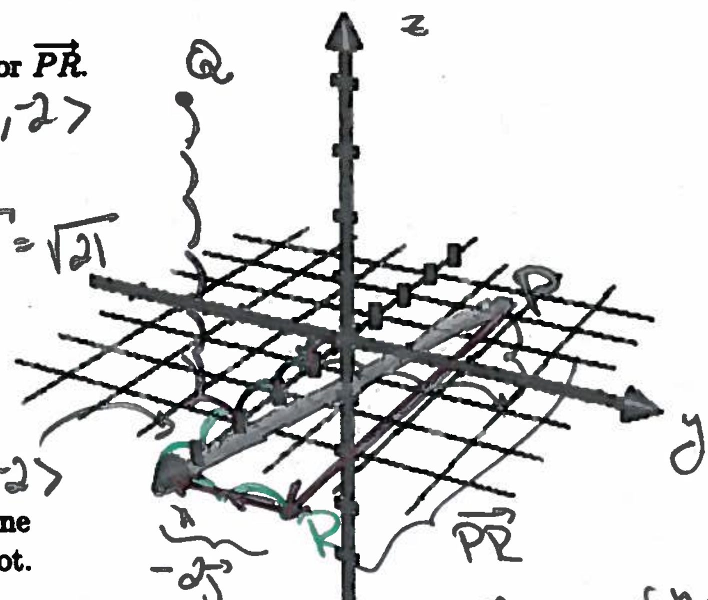
   (f) If f has a local minimum at (a,b) and f is differentiable at (a,b), then $\nabla f(a,b) = \vec{0}$.

   True this is the 3dimensional analogue of a Calc I Thm. In particular if we are at a max or min our partial derivatives will both evaluate to zero ⇒ $\nabla f(a,b) = \vec{0}$.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the points: $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$. Also let $S(3,6,1.5)$ and $T(-9,-14,-12.5)$.

   (a) Plot the points P, Q, and R.



   (b) Find the components of the vector $\overrightarrow{PR}$.

$\langle 5-1, 2-3, 0-2\rangle = \langle 4,-1,-2\rangle$

   (c) Find the length of $\overrightarrow{PR}$.

$\sqrt{4^2+(-1)^2+(-2)^2} = \sqrt{16+1+4} = \sqrt{21}$

   (d) Draw the vector $\overrightarrow{PR} - 2\vec{j}$ and then write its components.

$\langle 4,-1,-2\rangle - 2\langle 0,1,0\rangle$

$\langle 4,-1,-2\rangle + \langle 0,-2,0\rangle = \langle 4,-3,-2\rangle$

   (e) Use calculus methods to determine if ΔPQR is a right triangle or not.

note: we could have computed the length of each side + checked if the three lengths satisfied "$a^2+b^2=c^2$". Then the converse of the pythagorean thm would have given us the answer. But using calculus –

$\overrightarrow{PR} = \langle 4,-1,-2\rangle$
$\overrightarrow{PQ} = \langle 2,-4,4\rangle$
$\overrightarrow{QR} = \langle 2,3,-6\rangle$

we'll see if any of the 3 angles is 90° by using the dot product.

$\overrightarrow{PR}\cdot\overrightarrow{PQ} = 8+4-8 \neq 0$
$\overrightarrow{PR}\cdot\overrightarrow{QR} = 8-3+12 \neq 0$
$\overrightarrow{PQ}\cdot\overrightarrow{QR} = 4-12-24 \neq 0$

so there are no 90° angles.

   (f) Find the equation of the plane that passes through P, R, and Q.

I'll use the form $\vec{\eta}\cdot(\langle x,y,z\rangle - \langle x_1,y_1,z_1\rangle) = 0$
Note $\vec{\eta} \perp$ to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
the cross product would give me a $\perp$ vector.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \vec{i}(8-4) - \vec{j}(-4-16) + \vec{k}(-2-16)$$
$$= 12\vec{i} + 20\vec{j} + 14\vec{k}$$

so

$\langle 12,20,14\rangle\cdot(\langle x,y,z\rangle - \langle 1,3,2\rangle) = 0$

or (simplified)

$\langle 6,10,7\rangle\cdot(\langle x,y,z\rangle - \langle 1,3,2\rangle) = 0$

   (g) Does the line that passes through S and T intersect the plane you found in part (a)? Justify yourself.

Note $\overrightarrow{TS} = \langle 12, 20, 1.5+12.5\rangle$
$= \langle 12, 20, 14\rangle$

So the vector $\overrightarrow{TS}$ is $\parallel$ to the normal vector of the plane passing thru P, Q & R.

Not only will the line passing thru S & T intersect the plane, it will intersect it orthogonally.

6. [3] Consider the equation $2z = \frac{x^2}{2} - 2y^2$.

(a) Does the above equation describe a function of x and y? Why or why not?

Yup, each (x,y) returns only 1 z value

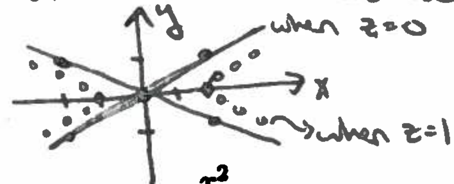
(b) Describe the contour curves of the graph of the equation above. That is, describe the intersection of the graph of the above equation with the planes $z = k$ where k is some constant.

if we set $z=k$ then

$2k = \frac{x^2}{2} - 2y^2$

$\Rightarrow 4k = x^2 - 4y^2$

So the contour curves look like hyperbola's.



$k=0 \Rightarrow (0,0)$
$(\pm 2, \pm 1)$

$k=1 \Rightarrow (2,0)$

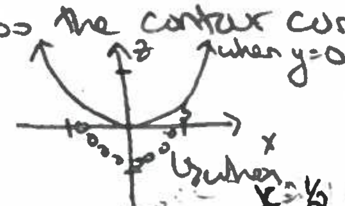
(c) Describe the intersection of the graph of $2z = \frac{x^2}{2} - 2y^2$ with planes parallel to the xz axis. That is, when $y = k$ for some constant k.

if we set $y=k$ then

$2z = \frac{x^2}{2} - 2k^2$

$\Rightarrow z = \frac{x^2}{4} - k^2$

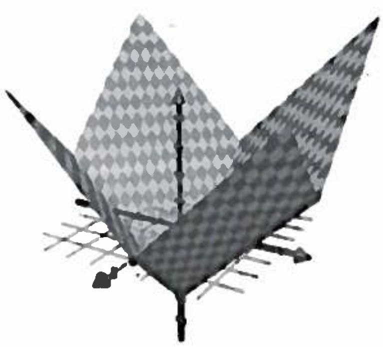
So the contour curves look like parabola's opening up

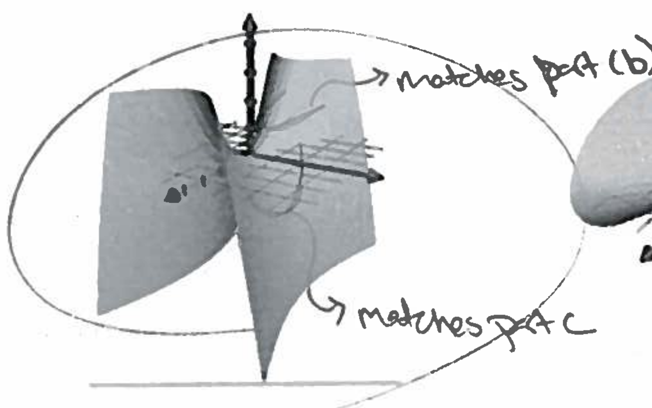


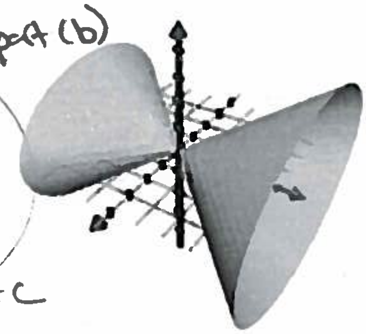
$k=0 \Rightarrow (1, \frac{1}{4})$

$k=1/2 \Rightarrow (1, -.75)$

(d) Which (if any) of the following is a graph of the above function?



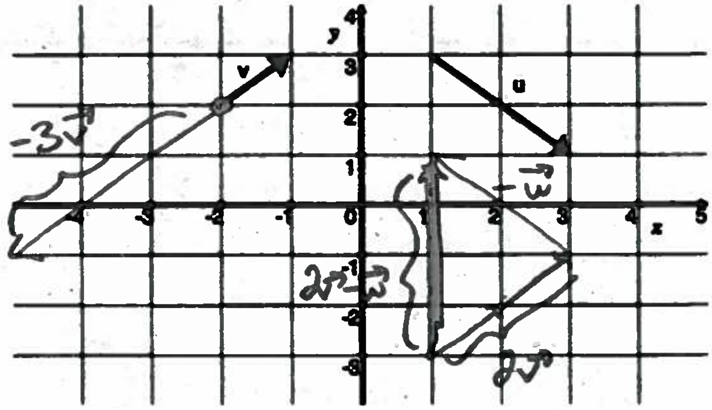




note if $x=k$ for some constant.

$2z = \frac{k^2}{2} - 2y^2$ we have parabolas opening down which also agree with this middle graph

4. **Consider the vector $\vec{v}$ and $\vec{u}$ shown to the right.**



(a) **Draw the vector $-3\vec{v}$.**

(b) **Draw the vector $2\vec{v} - \vec{u}$.**

(c) **Find the projection of $\vec{u}$ onto $\vec{v}$.**

These looks ⊥ to $\vec{v}$...

$\vec{w} \cdot \vec{v} = \langle 2,-2 \rangle \cdot \langle 1,1 \rangle$

$= 2 \cdot 1 + (-2)(1) = 0$

So yeah, perpendicular

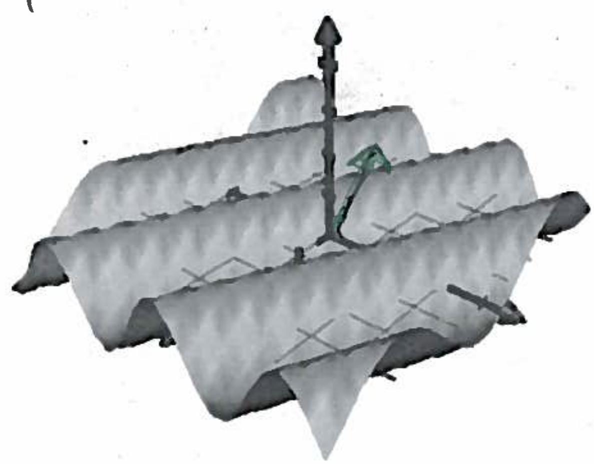
So the projection of $\vec{w}$ onto $\vec{v}$ would be $\vec{0}$

done another way

$\text{proj}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$

$= \frac{0}{\sqrt{2}^2} \langle 1,1 \rangle = \vec{0}$

5. **Consider the function $f(x, y) = -\sin(x + 2y)$ for the following questions.**



(a) **Find the gradient of f.**

$\langle -\cos(x+2y), -2\cos(x+2y) \rangle$

(b) **Evaluate the gradient at the point $(0, 0)$.**

$\langle -\cos(0+2 \cdot 0), -2\cos(0+2 \cdot 0) \rangle = \langle -1, -2 \rangle$

(c) **Interpret your answer in (b) graphically and consider referencing the graph of f shown to the right.**

When at the origin, the direction of the steepest assent is in the direction of $\langle -1,-2 \rangle$

(Drawn on the graph above)

(d) **[3] Find the linear approximation of f at the point $(0, 0)$.**

I'll use $z - z_1 = m_x(x - x_1) + m_y(y - y_1)$

from (b) $m_x = f_x(0,0) = -1$

$m_y = f_y(0,0) = -2$

b/c the function passes thru $(0,0,-\sin(0))$

$z - 0 = -1(x-0) - 2(y-0)$

9. **Use Calculus methods to to find the (x, y, z) coordinates in $\mathbb{R}^3$ to find and classify the critical points of the function**

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4.$$

note: a graph of this function would normally be provided.

$f_x(x,y) = 20xy - 10x - 4x^3$
$f_y(x,y) = 10x^2 - 8y - 8y^3$

since I don't have a graph, I'll have to use the 2nd der test so I'll compute those:

$f_{xx}(x,y) = 20y - 10 - 8x$
$f_{xy}(x,y) = 20x$
$f_{yy}(x,y) = -8 - 24y^2$

finding the critical Points:

$f_x(x,y)=0$, $f_y(x,y)=0$ $\Rightarrow$ $20xy - 10x - 4x^3 = 0$, $10x^2 - 8y - 8y^3 = 0$ $\Rightarrow$ $2x(10y - 5 - 2x^2) = 0$ (*), $10x^2 - 8y - 8y^3 = 0$ (+)

by (*) $2x = 0$ or $10y - 5 - 2x^2 = 0 \Rightarrow x^2 = \frac{1}{2}(10y - 5)$ (☺)

$\Rightarrow x = 0$

$\Rightarrow 10 \cdot 0^2 - 8y - 8y^3 = 0$ by (+)

$\Rightarrow -8y(1 + y^2) = 0$

So $8y = 0$ or $1 + y^2 = 0$

$\Rightarrow y = 0$ no real sol

$\therefore (0,0)$ is a critical point

sub into (+)

$\Rightarrow 10(\frac{1}{2}(10y - 5)) - 8y - 8y^3 = 0$

$\Rightarrow 50y - 25 - 8y - 8y^3 = 0$

$\Rightarrow -8y^3 + 42y - 25 = 0$

cubic we need to find the roots - - Newton's method?

(look on next page for this)
(you could have used your calc. for this as well...)

$y = .6468$ $y = -2.5452$ $y = 1.8994$

$\Rightarrow x = \pm .8567$ no real sol. $x = \pm 2.6442$

by (☺)

using $D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D(0,0) = -10 \cdot -8 - (8)^2 > 0$ and $f_{xx}(0,0) = -10 < 0$ } local max at $(0,0)$

$D(.8567, .6468) = -187.64 < 0$ saddle

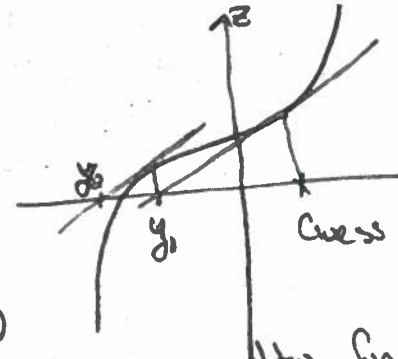
$D(-.8567, .6468) = -.87.64 < 0$ saddle

$D(2.6442, 1.8994) \approx 2500 > 0$; $f_{xx} < 0$ } local max

$D(-2.6442, 1.8984) \approx 2500 > 0$; $f_{xx} < 0$ } local max

Given $F(y) = -8y^3 + 42y - 25$ we need to find the roots (looking at end behavior we know there is at least one)

Recall Newton's method uses the tangent lines to approx roots (picture to the right)



tangent line at y_0:

$$z - F(y_0) = F'(y_0)(y - y_0)$$

tangent line zeros out when $z = 0$, thus we want to find y ∋

$$0 - F(y_0) = F'(y_0)(y - y_0)$$

$$\Rightarrow \frac{-F(y_0)}{F'(y_0)} = y - y_0 \quad \Rightarrow y = y_0 - \frac{F(y_0)}{F'(y_0)}$$

note: if you remembered that formula - you could have written it down - I didn't have it written anywhere close!

So $y = y_0 - \frac{F(y_0)}{F'(y_0)}$ where $F(y) = -8y^3 + 42y - 25$
$F'(y) = -24y^2 + 42$

Guess $y_0 = 0$
$\Rightarrow y_1 = .5952$
$\Rightarrow y_2 = .6456$
$\Rightarrow y_3 = .6468$
$\Rightarrow y_4 = .6468$

Guess $y_0 = -2$
$\Rightarrow y_1 = -2.8333$
$\Rightarrow y_2 = -2.5814$
$\Rightarrow y_3 = -2.5458$
$\Rightarrow y_4 = -2.5452$
$\Rightarrow y_5 = -2.5452$

Guess $y_0 = 2$
$\Rightarrow y_1 = 1.9074$
$y_2 = 1.8985$
$y_3 = 1.8984$
$y_4 = 1.8984$

note: after finding one root I could have done long division:

$$\begin{array}{r} -8y^2 - 5.1744y + 38.6532 \\ y - .6468 \overline{)\, -8y^3 + 42y - 25} \\ -(-8y^3 + 5.1744y^2) \\ \hline -5.1744y^2 + 42y - 25 \\ -(-5.1744y^2 + 3.3468y) \\ \hline \end{array}$$

to find the quadratic + thus find the roots that way.

$$\Rightarrow y = \frac{5.1744 \pm \sqrt{5.1744^2 - 2(-8)(38.6532)}}{2(-8)}$$