Exam 1

Tmath 126

Practice

Key

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   (a) If there exists some number M such that $a_n \leq M$ for all n, then $\{a_n\}$ converges.

   False. Let $a_n = (-1)^n$ then $a_n \leq 1$ for all n but $\{-1, 1, -1, 1, -1, \ldots\}$ does not converge

   (b) Every point on the complex plane has a set of polar coordinates which are unique.

   False. Note $1e^{i \cdot 0} = 1e^{2\pi i} = 1e^{4\pi i} = 1e^{6\pi i} = \ldots$ there are an infinite # of polar coordinates to describe 1.

   (c) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

   True. Consider a logically equivalent statement (contrapositive) If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge. Which is definitely true

   (d) The Taylor series is an example of a power series.

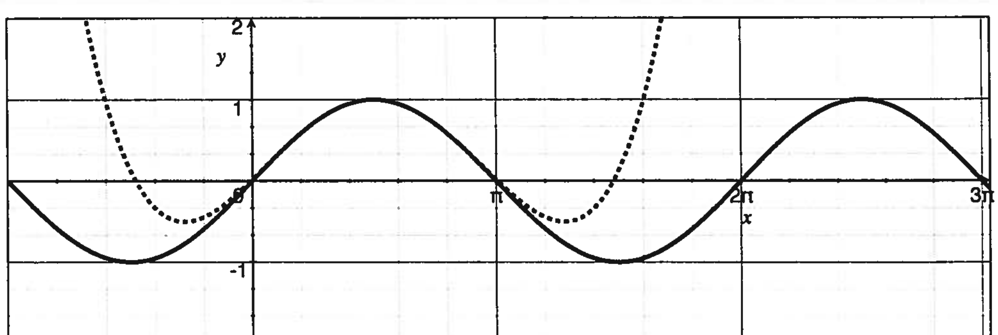
   True. A power series is any kind of 'infinite polynomial' which the Taylor series is.

   (e) Given a function f, the associated taylor series T has the property that $f(x) = T(x)$ for all x.

   False. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$ but only when x is between -1 + 1.

   (f) The dotted function below is the 4$^{\text{th}}$ Taylor polynomial of $\sin(x)$ centered at 0.

False It looks like the (dotted) 4^{th} deg poly. is centered around $\pi/2$ + not zero



Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Write the following sum in expanded form and simplify: $\sum_{n=1}^{4} \frac{\sqrt{2n+1}}{n!}$

$$\frac{\sqrt{2(1)+1}}{1!} + \frac{\sqrt{2(2)+1}}{2!} + \frac{\sqrt{2(3)+1}}{3!} + \frac{\sqrt{2(4)+1}}{4!} = \frac{\sqrt{3}}{1} + \frac{\sqrt{5}}{2} + \frac{\sqrt{7}}{6} + \frac{\sqrt{9}}{24}$$

$$= \sqrt{3} + \sqrt{5}/2 + \sqrt{7}/6 + 1/8$$

3. Write the following sum using the sigma notation: $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81}$

$$\sum_{i=0}^{4} \frac{i+1}{3^i}(-1)^i$$ works or $$\sum_{n=1}^{5} \frac{n}{3^{n-1}}(-1)^{n+1}$$

(note: there are many other possible right answers)

4. Compute the following if possible.

$$\lim_{n\to\infty} \frac{10^{n+1}}{9^n} = \lim_{n\to\infty} \frac{10^1 10^n}{9^n}$$

$= 10 \lim_{n\to\infty} \left(\frac{10}{9}\right)^n$ since $10/9 > 1$

$(10/9)^n \to \infty$ as $n \to \infty$

thus $\lim_{n\to\infty} \frac{10^{n+1}}{9^n}$ diverges

$$\sum_{n=1}^{\infty} \frac{(2e)^n}{6^{n-1}} = \frac{2e}{1} + \frac{(2e)^2}{6} + \frac{(2e)^3}{6^2} + \frac{(2e)^4}{6^3} + \dots$$

looks like a geometric series

$a + ar + ar^2 + ar^3 + ar^4 + \dots$

where $a = 2e$ and $r = 2e/6 = e/3$.

Worksheet in class

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(2e)^n}{6^{n-1}} = \frac{a}{1-r} = \frac{2e}{1-e/3} = \frac{6e}{3-e}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{0!} + \frac{-1}{1!} + \frac{1}{2!} + \frac{-1}{3!} + \dots$$

This reminds me of the Taylor series of $e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

If we let $x = -1$ in the above series we'd have

$$\frac{1}{0!} + \frac{1}{1!}(-1) + \frac{1}{2!}(-1)^2 + \frac{1}{3!}(-1)^3 + \dots$$

which is the series we were considering so $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}$

$$\sum_{n=1}^{\infty} \frac{2}{n} = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \dots$$

$$= \underbrace{2\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\right)}_{\text{harmonic series which we know diverges}}$$

or

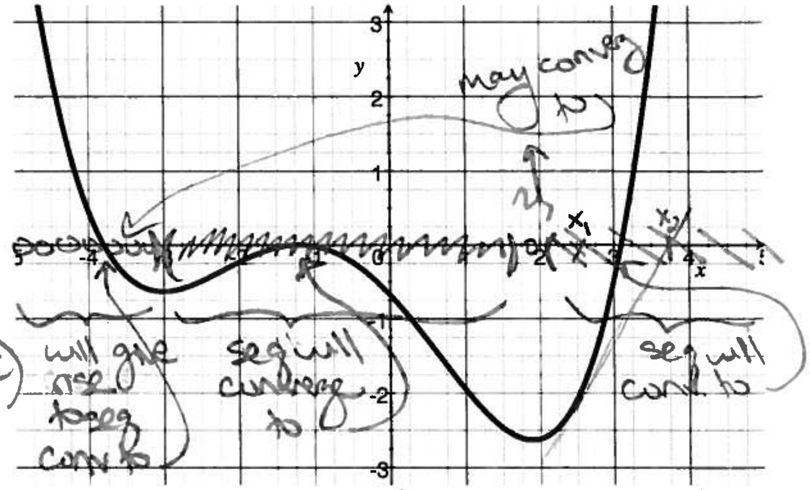
$$2\left[1 + \frac{1}{2} + \boxed{\frac{1}{3} + \frac{1}{4}} + \boxed{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} + \dots\right]$$

$$\geq 2\left[1 + \frac{1}{2} + \boxed{\frac{1}{4} + \frac{1}{4}} + \boxed{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} + \dots\right]$$

$$= 2\left[1 + \frac{1}{2} + \boxed{\frac{1}{2}} + \boxed{\frac{1}{2}} + \dots\right]$$

which diverges

5. Let $p(x) = 0.04 * x^4 + 0.124 * x^3 - 0.3464 * x^2 - 1.09056 * x - 0.678528$ whose graph is shown to the right.



$p'(x) = .16x^3 + .372x^2 - .6928x - 1.09056$

(a) If you wanted to use Newton's method to find the positive root of the function p, what would your first guess be (x_1)?

2.5 should work

(note: any # between 2 & ∞ should work)

(b) Using x_1 you choose in part (a), use Newton's method to find x_2.

Know

$$x_2 = x_1 - \frac{p(x_1)}{p'(x_1)}$$

$$= 2.5 - \frac{-2.0699}{2.0024}$$

$$= 3.5337$$

or if you are like me...

find x-intercept of line tangent to p when x=2.5

tangent line $y = mx + b$ $\quad m = p'(2.5) = 2.0024$

pass thru $(2.5, -2.0699)$

$\Rightarrow y + 2.0699 = 2.0024(x - 2.5)$

x-intercept @ $y=0$ so solve for x $\rightsquigarrow x = 3.5337$

(c) Identify the basins of convergence with Newton's method on the graph.

(d) Find the second order Taylor polynomial $T_2(x)$ based at $b = 1$.

$$\frac{p(1)}{0!} + \frac{p'(1)}{1!}(x-1) + \frac{p''(1)}{2!}(x-1)^2 = -1.9515 - 1.2514(x-1) + \frac{.5312}{2}(x-1)^2$$

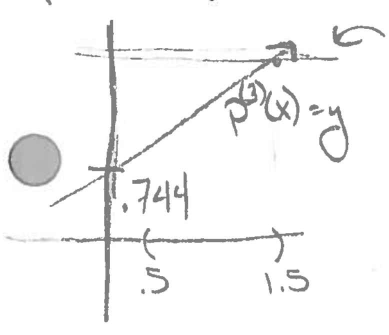
i	$p^{(i)}(x)$	$p^{(i)}(1)$
0	see above	-1.9515
1	$.16x^3 + .372x^2 - .6928x - 1.09056$	-1.2514
2	$.48x^2 + .744x - .6928$	.5312

(e) Bound the error $|p(x) - T_2(x)|$ on the interval [0.5,1.5].

we need to find M so that $p^{(3)}(x) \le M$ for x between .5 & 1.5

$p^{(3)}(x) = .96x + .744$

Let $M = 2.2$ (note: many other choices would have worked)



M above this y value would work

$p(1.5) = 2.1840$

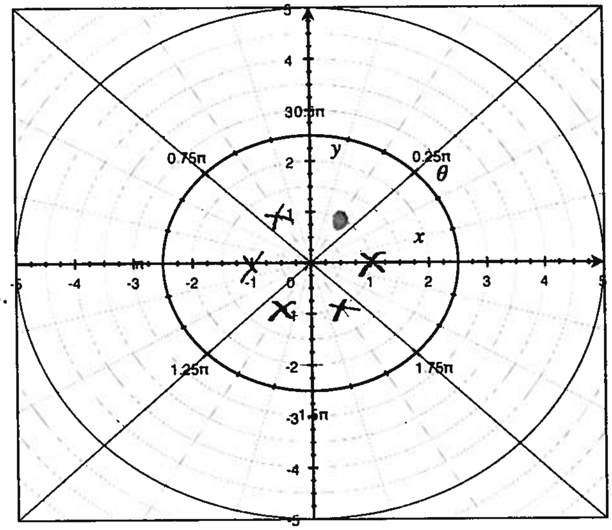
$$|p(x) - T_2(x)| < \frac{M}{3!}(1.5-1)^3$$

$$= \frac{2.1840}{6}(.5)^3$$

$$= .0455$$

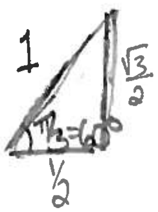
6. Consider the point $Q = e^{\frac{i13\pi}{3}}$

(a) Simplify Q so that the angle is between 0 and 2π and then plot Q on the axis provided.



$\frac{13\pi}{3} = \frac{12\pi}{3} + \frac{\pi}{3} = 4\pi + \frac{\pi}{3} \Rightarrow Q = e^{\frac{i\pi}{3}}$

(b) Write Q with rectangular coordinants.



Sohcahtoa

$$\left.\begin{aligned} \sin \tfrac{\pi}{3} &= \tfrac{y}{1} \\ \cos \tfrac{\pi}{3} &= \tfrac{x}{1} \end{aligned}\right\} \quad \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

(c) Find $Q^6 - 1$.

$$Q^6 - 1 = \left(e^{\frac{i\pi}{3}}\right)^6 - 1 = e^{\frac{6\pi}{3}i} - 1 = e^{2\pi i} - 1 = 1 - 1 = 0$$

(d) Find two solutions of $x^6 = 1$.

1 and (from above) $e^{i\pi/3}$ oh, I guess -1 would work

(e) ExtraCredit: The Fundamental Theorem of Algebra implies there are six solutions of $x^6 = 1$. Find the other 4. These are the sixth roots of unity?

I've denoted the other solution with x's on the above graph.

Besides I marked $e^{\frac{2\pi i}{3}}, e^{\frac{3\pi i}{3}}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}, e^{\frac{6\pi i}{3}}$

or $e^{\frac{2\pi i}{3}}, -1, e^{4\pi i/3}, e^{5\pi i/3}, 1$

7. Find the Taylor series expansion for $\frac{x}{4+x}$ centered at 0, and find out where it converges.

Rather then work from scratch (which would work) I'll use

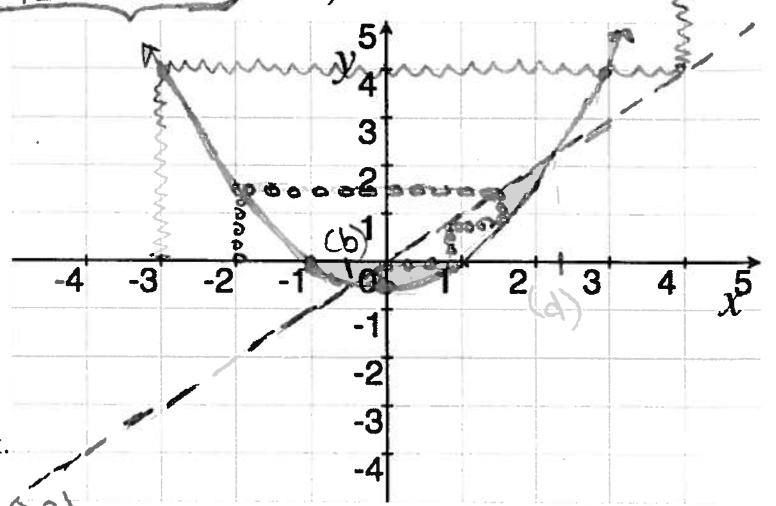
$\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \dots$ which converges as long as $u \in (-1,1)$

$$\text{So } \frac{x}{4+x} = \frac{x}{4}\,\frac{1}{1+x/4} = \frac{x}{4}\,\frac{1}{1-\left(\frac{-x}{4}\right)} = \frac{x}{4}\left(1 + \frac{-x}{4} + \left(\frac{-x}{4}\right)^2 + \left(\frac{-x}{4}\right)^3 + \left(\frac{-x}{4}\right)^4 + \dots\right)$$

$$= \frac{x}{4}\left(1 - \frac{x}{4} + \frac{x^2}{4^2} - \frac{x^3}{4^3} + \frac{x^4}{4^4} - \dots\right) = \frac{x}{4} - \frac{x^2}{4^2} + \frac{x^3}{4^3} - \frac{x^4}{4^4} + \dots = \sum_{i=1}^{\infty} \left(\frac{x}{4}\right)^i (-1)^{i+1}$$

Converges as long as $-1 < -\frac{x}{4} < 1 \Rightarrow -4 < -x < 4$ or $4 > x > -4$

8. Consider the recursively defined sequence
$\{a_1, \boxed{\tfrac{1}{2}(a_1^2-1)}, \tfrac{1}{2}\left(\left(\boxed{\tfrac{1}{2}(a_1^2-1)}\right)^2-1\right), \tfrac{1}{2}\left(\tfrac{1}{2}\left(\left(\boxed{\tfrac{1}{2}(a_1^2-1)}\right)^2-1\right)^2-1\right), \ldots\}$.



(a) Identify the recursive function $R(x)$ so that $a_{n+1} = R(a_n)$.

$\frac{1}{2}(x^2-1)$ Parabola
vert shift down 1
vert shrunk by $\frac{1}{2}$

(b) If $a_1 = -2$, does the resulting sequence converge? If so, identify what it converges to (either find the number or identify it on a graph). Be sure to show your work.

Converges to x coord. denoted with (b).
(used cobwebbing oooo path)

(c) If $a_1 = -3$, does the resulting sequence converge? If so, identify what it converges to (either find the number or identify it on a graph). Be sure to show your work.

Diverges to ∞
(used cobwebbing path)

(d) How many different limits can the sequence of a_n converge to? Justify yourself.

There are only two finite limits that exist (b) + (d) that correspond to where $R(x)$ intercepts $y=x$. The cobwebbing procedure (which follows a seq.) will approach one of these 2 locations or diverge

9. Use geometric series to show $0.99999\ldots = 1$.

Note $.\overline{999} = .9 + .09 + .009 + .0009 + \ldots$

$= 9\frac{1}{10} + 9\left(\frac{1}{10}\right)^2 + 9\left(\frac{1}{10}\right)^3 + 9\left(\frac{1}{10}\right)^4 + \ldots$

which looks like a geometric series $a + ra + r^2a + r^3a + \ldots$
where $a = 9/10$ and $r = 1/10$.

So the series converges to $\frac{a}{1-r} = \frac{9/10}{1-1/10} = \frac{9/10}{9/10} = 1$

Thus $.\overline{999} = 1$. Cool huh?? We can write the #1 in a dif way!