(+1) report collaboration

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1. ~~[8]~~ TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   (a) For any vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ in $\mathbb{R}^3$, $||\overrightarrow{v} + \overrightarrow{w}|| = ||\overrightarrow{v}|| + ||\overrightarrow{w}||$.

False (+1)

Consider $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Notice $||\vec{v} + \vec{w}|| = ||\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}|| = ||\begin{bmatrix} 0 \\ 0 \end{bmatrix}|| = 0$

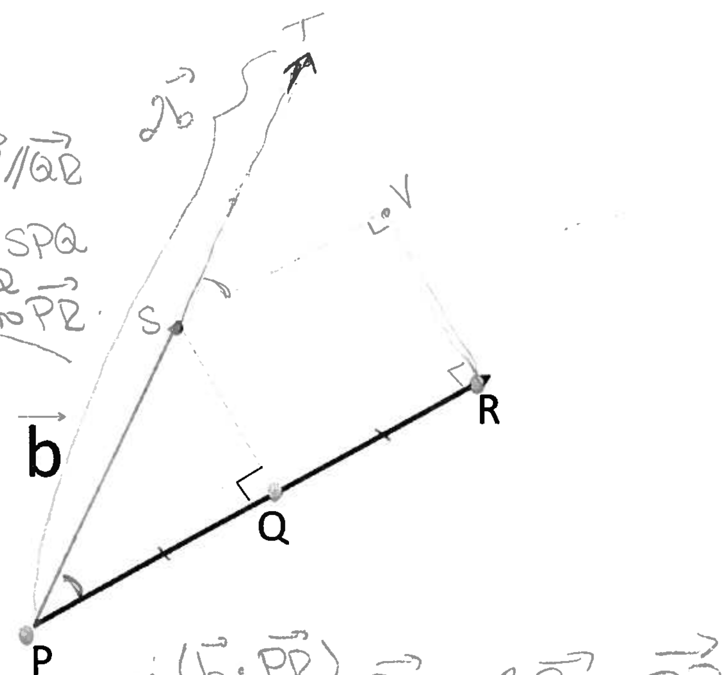
but $||\vec{v}|| + ||\vec{w}|| = ||\begin{bmatrix} 1 \\ 0 \end{bmatrix}|| + ||\begin{bmatrix} -1 \\ 0 \end{bmatrix}|| = 1 + 1 = 2$

and $0 \neq 2$.

start (+.5)
reasoning/sense (+1)
algebra/arithmetic (+.5)
found countr ex (+1)
.5 collaboration

   (b) If the points P, Q, R and the vector $\overrightarrow{b}$ are as shown, and $||\overrightarrow{PQ}|| = ||\overrightarrow{QR}||$, then $\text{proj}_{\overrightarrow{PR}} 2\overrightarrow{b} = \overrightarrow{PR}$.

Since $\overrightarrow{SV} // \overrightarrow{QR}$ we know $\angle TSV \cong \angle SPQ \Rightarrow \triangle TSV \cong \triangle SPQ$
~~So $\overrightarrow{PT}$ projects onto $\overrightarrow{PR}$~~



True (+1)

.5 collaboration
start (+.5)
sense (+1)

Notice that by the definition of projection

$\text{proj}_{\overrightarrow{PR}} \vec{b} = \overrightarrow{PQ}$.

(+1.5) Since

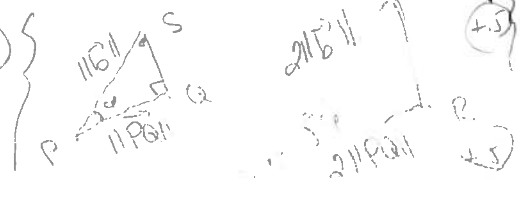
$$\text{proj}_{\overrightarrow{PR}} 2\vec{b} = \frac{2\vec{b} \cdot \overrightarrow{PR}}{\overrightarrow{PR} \cdot \overrightarrow{PR}} \overrightarrow{PR} = 2 \frac{(\vec{b} \cdot \overrightarrow{PR})}{\overrightarrow{PR} \cdot \overrightarrow{PR}} \overrightarrow{PR} = 2\overrightarrow{PQ} = \overrightarrow{PR}$$

(by dot properties)

or

Consider geometrically $2\vec{b}$ shown above notice triangles $\triangle PQS$ and $\triangle PRT$ the law of cosines & the converse of the Pythagorean theorem imply the 2 $\triangle$s are similar $\Rightarrow \angle R$ is $90°$

(+.5)



(+.5) $\Rightarrow \text{proj}_{\overrightarrow{PR}} 2\vec{b} = \overrightarrow{PR}$.

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Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2 pt collaboration (+1)

2. 7 [~~4~~] You are given the following data of a function $g(x,y)$. Your boss wants you to approximate $g(.8, 1.4)$ and wants to be convinced you're doing something sophisticated. Find a linear approximation for your boss and explain your choices (there are many that you will make!).

(+1) We want to find the equation of a plane using the three points closest to $x = .8$ & $y = 1.4$. I've identified the three.

	x	y	$g(x,y)$	Δx	Δy
P	0.55	1.2	27	.25	.2
	0.65	1.0	31	.15	.4
Q	0.65	1.1	29	.15	.3
R	0.75	1.2	50	.05	.2

(+4) Finding an Equation of the Plane eq of plane +1

+1 vectors $\vec{PQ} = [.1, -.1, 2]$, $\vec{PR} = [.2, 0, 23]$

+1.5 normal

$$\begin{vmatrix} .1 & -.1 & 2 \\ .2 & 0 & 23 \end{vmatrix} = \vec{i}(-2.3) - \vec{j}(2.3 - .4) + \vec{k}(+.02) = [-2.3, 1.9, .02]$$

+.5 So $0 = \langle -2.3, 1.9, .02\rangle \cdot (\langle x, y, z\rangle - \langle .55, 1.2, 27\rangle)$

(+1) An estimate is $x = .8$ & $y = 1.4$ is about ≈ 36.75

repeat collaboration (+1)

3. An ant infected by the myrmeconema neutroncum parasite wants to climb as high as possible to make it more likely to be eaten by birds. The algorithm is pretty simple: go in the direction of the steepest assent. The ant is at $(1, 2, \cos(2))$ and climbing on the surface described by $h(x,y) = \cos(xy)$.

(a) 5.5 [~~5~~] If the ant travels in the direction of $\langle -1, 3\rangle$, what is the approximate change in elevation experienced by the ant? Be sure to provide work so I can see where your numbers come from.

(+.5) We'd like to know the approximate change in z or directional derivative

(+1) note $\|\langle -1, 3\rangle\| \neq 1$ so use $\frac{1}{\sqrt{1+9}}\langle -1, 3\rangle$ or $\langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\rangle$ (+1)

(+.5) $D_{\vec{u}}(a,b) = \vec{u} \cdot \nabla f(a,b) = \langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\rangle \cdot \langle -y\sin(xy)\big|_{(1,2)}, -x\sin(xy)\big|_{(1,2)}\rangle$ (+1)

alg/arithmetic (+.5)

$= \langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\rangle \langle -2\sin(2), -\sin(2)\rangle$

(+.5) $= \frac{2}{\sqrt{10}}\sin(2) - \frac{3}{\sqrt{10}}\sin(2) \approx -.288$

(b) 3.5 [~~3~~] Which direction does the ant want to travel?

(+1) The ant wants to go in the direction of the steepest assent or ∇ (+1)

know ∇ (+.5)

$\nabla f(1,2) = \langle -y\sin(xy)\big|_{(1,2)}, -x\sin(xy)\big|_{(1,2)}\rangle$

$= \langle -2\sin(2), -\sin(2)\rangle$ (+.5)

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(+1) report collaboration

4. Consider the sphere S centered at $(1, 2, 3)$ with radius 4.

(a) [2] Write an algebraic equation for the sphere S.

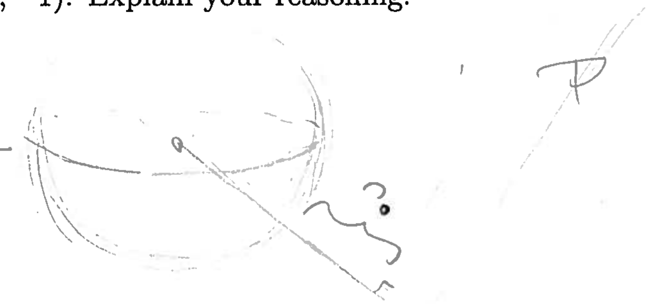
$(x-1)^2+(y-2)^2+(z-3)^2=16$

or $z = 3 \pm \sqrt{16-(x-1)^2-(y-2)^2}$

subtract center (+1)
radius (+.5)
from (+.5)

[10] (b) ~~[6]~~ Find the shortest distance between the sphere S and the plane passing through the points $(0, 2, -2)$, $(-1, -1, -2)$ and $(4, 4, -1)$. Explain your reasoning.

(+2) Explain reasoning/sense
(+1) Algebra/arithmetic



note the shortest distance from a plane P is when the plane tangent to the sphere is // to P.

(+3) finding the equation of P
+1 vectors in plane: $\langle 1,3,0\rangle$ & $\langle 4,2,1\rangle$
+1 cross product: $\langle 3,-1,-10\rangle$
+1 eq of plane: $\langle 3,-1,-10\rangle(\langle x,y,z\rangle-\langle 0,2,-2\rangle)=0$

or $2+\frac{3}{10}(x-0)-\frac{1}{10}(y-2)=z$

We want to find (a,b,c) so that

(+1) $\{ g_x(a,b)=3/10$ AND $g_y(a,b)=-1/10$ or

where $g(x,y)=3\pm\sqrt{16-(x-1)^2-(y-2)^2}$

i.e. the two planes have equal slopes

(+1) (+1)

$\frac{-2(x-1)}{\sqrt{16-(x-1)^2-(y-2)^2}}=3/10$ AND $\frac{-2(y-2)}{\sqrt{16-(x-1)^2-(y-2)^2}}=-1/10$

$\Rightarrow 4(x-1)^2=\frac{9}{100}(16-(x-1)^2-(y-2)^2)$

$\Rightarrow (x-1)^2=\frac{100}{409}\left[\frac{9.4}{25}-\frac{9}{100}(y-2)^2\right]$

substitute into 2nd eq

algebra (.5)

$\frac{-2(y-2)}{\sqrt{16-\frac{100}{409}\left[\frac{36}{25}-\frac{9}{100}(y-2)^2\right]-(y-2)^2}}=-1/10$

(+.5) Solving for y then x gives us the points on the sphere closest to the plane then we can compute the distance to plane

note the shortest distance from a plane is along the line normal to the plane that passes thru the center of the circle

(+3) finding the equation of the plane.
+1 vectors in plane: $\langle 1,3,0\rangle$ & $\langle 4,2,1\rangle$
+1 eq. of plane $\langle 3,-1,-10\rangle(\langle x,y,z\rangle-\langle 0,2,-2\rangle)=0$
+1 cross product: $\langle 3,-1,-10\rangle$
$\vec{i}(3-0)-\vec{j}(1-0)+\vec{k}(2-12)$

(+1) finding line thru center w/ dir

$(x,y,z)=(1,2,3)+t\langle 3,-1,-10\rangle$

(+2) finding where line intersects plane

$\langle 3,-1,-10\rangle(\langle 1+3t,2-t,3-10t\rangle-\langle 0,2,-2\rangle)=0$

$\Rightarrow t=\frac{-47}{110}$

So @ $(2.28, 1.57, -1.27)$

(+.5) length of segment from plane to center $\sqrt{1.28^2+.43^2+4.27^2}$

(+.5) length from plane to sphere (subtract 4) $\approx .47$ 13

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5. ~~[8]~~ Role play that you are a mathematician in the 18th century. Another mathematician has contacted you bragging/challenging you to solve the following problem: Identifying all the critical points for

$$f(x,y) = x^4 + 2x - 6xy + 2y + y^4$$

report (+1) collaboration

and classify each as a minimum, maximum or neither. Write back with your solution.

Note: your audience is another mathematician of a similar level as you, the numbers are not nice, and the graphing calculator is not going to be available for a few more centuries...

(+1) appropriate audience

(+.5) Critical Points are all (a,b) so that $f_x(a,b)=0$ & $f_y(a,b)=0$

(+.5) $f_x(x,y) = 4x^3 + 2 - 6y$

(+.5) $f_y(x,y) = -6x + 2 + 4y^3$

$f_{xx}(x,y) = 12x^2$ (+.5)

$f_{yy}(x,y) = 12y^2$ (+.5)

$f_{xy}(x,y) = -6$ (+.5)

single variable (+1)

I've included my various derivatives for easy reference.

Finding the Critical Points:

$4x^3 + 2 - 6y = 0 \Rightarrow y = \frac{1}{6}(4x^3+2)$ so substituting we have

$-6x + 2 + 4y^3 = 0$

$g(x) = -6x + 2 + 4\left[\frac{1}{6}(4x^3+2)\right]^3 = 0$

which we can distribute & combine like terms to get.

Newton +.5, right +1, y +.5 (+2)

To find the roots I'm going to use Newton's new method!

my initial guess: $x_0 = 0$ which lead to $x_1 = .3580$, $x_2 = .366 = x_3$

(using of course that x_{i+1} is the y intercept of the line tangent to $g(x)$ at x_i or $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$)

another guess led to -1.366 of course 1 is a root.

explain/sense (+1)

If $x=1$ then $0 = 4 + 2 - 6y \Rightarrow y = 1$

If $x = .366$ then $0 = 6(.366) + 2 + 4y^3 \Rightarrow y = .366$

If $x = 1.366$ then $y = -1.366$

(+.5) Our Critical Points are thus

$(1, 1, f(1,1))$ AND $(.366, .366, f(.366))$ AND $(-1.366, -1.366, f(-1.366, -1.366))$

(+1) Now I'll just run my 2nd derivative test to identify each

(+1.5)

$f_{xx}(1,1)f_{yy}(1,1) - [f_{xy}(1,1)]^2 > 0$, $f_{xx} > 0$ $\Rightarrow (1,1,f(1,1))$ is a Minimum

$f_{xx}(.366,.366)f_{yy}(.366,.366) - [f_{xy}(.366)]^2 < 0$ $\Rightarrow (.366, .366, f(.36, .36))$ is a Saddle

$f_{xx}(-1.366,-1.366)f_{yy}(-1.36,-1.36) - [f_{xy}(\ )]^2$, $f_{xx} > 0$ $\Rightarrow (-1.366, -1.366, f(-1.36, -1.36))$ is a Minimum

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