Key

Exam 1 — **Tmath 126** — Summer 2014

1. [12] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   (a) (§11.1 #84 Cobwebbing) Let a_n be a recursively defined sequence where $a_1 = 4$, $a_n = f(a_{n-1})$, and f is a continuous function. If $\lim_{n\to\infty} a_n = 6$, then $f(6) = 6$.

   stat (.5)
   reasoning/sense (1)
   technique/cobweb (1)

   True. (.5) We can "see" the limits visually by graphing the function f, the line y=x, and using the cobwebbing approach. Limits of sequences correspond with points where f intersects y=x which is exactly what f(6)=6 would be saying

   (b) (PracticeExam #1) Given a function f, the associated taylor series T has the property that $f(x) = T(x)$ for all x.

   stat (.5)
   looking for ex (1)
   reasoning sense (1)

   False (.5) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$

   but only when x is between -1 and 1

   (c) (pg778 #1) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

   stat (.5)
   looking for ex (1)
   reasoning/sense (1)

   False (.5) Consider $\sum_{n=1}^{\infty} \frac{1}{n}$ i.e. the harmonic series

   Notice $\lim_{n\to\infty} a_n = 0$ but $\sum_{n=1}^{\infty} 1/n$ diverges to ∞

   (d) (Lecture 7/8) The first degree Taylor polynomial of a function f centered at 2 is the same as the line tangent to f when x=2.

   stat (.5)
   reasoning/sense (1)
   definitions (1)

   True (.5) by construction, the first Taylor polynomial is such that $T_1(2) = f(2)$ and $T_1'(2) = f'(2)$ which gives rise to the line tangent to f @ $x=2$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the series: $\frac{1}{1} + \frac{-1}{1} + \frac{1}{2} + \frac{-1}{6} + \frac{1}{24} + \frac{-1}{120} + \ldots$

2·1   3·2·1   4·3·2·1   5·3·2·1

(a) [3] (WebHW5 #1) Write the series using sigma notation:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \quad +1 \quad +1$$

+1

(b) [3] (PracticeExam #4) Determine what the series above converges to, if it converges. Justify your work.

try to convince of convergence +.5

notice $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ which looks a lot like (a) +1

but $x=-1$ so $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1}$ +.5

+1

Note: e^x converges to $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x so certainly when $x=-1$

3. Compute the following if possible.

(a) [4] (§11.1 #30) $\lim_{n\to\infty} a_n$ where $a_n = \sqrt{\frac{n+2}{25n-1}}$

a_n +.5

$$\lim_{n\to\infty} \sqrt{\frac{n+2}{25n-1}} \overset{+1}{=} \sqrt{\lim_{n\to\infty} \frac{n+2}{25n-1}} = \sqrt{\lim_{n\to\infty} \frac{n}{25n}}$$

b/c dominated by leading terms as $n\to\infty$ +1

$$= \sqrt{\frac{1}{25}} = \frac{1}{5}$$

+.5

+1

(b) [4] (WebHW4 #4) The series $\sum_{n=0}^{\infty} a_n$ where $a_n = 6(0.1)^n$

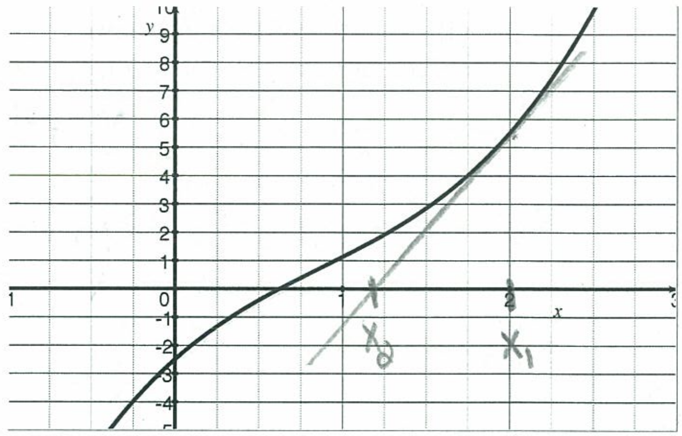
$$\sum_{n=0}^{\infty} a_n \overset{+.5}{=} \sum_{n=0}^{\infty} 6(0.1)^n = 6(.1)^0 + 6\cdot(.1)^1 + 6\cdot(.1)^2 + 6\cdot(.1)^3 + \ldots \quad +.5$$

geometric series +1

$a=6$ +.5 $\quad r=.1$ +.5

converges to $\frac{6}{1-.1}$ +1 or $\frac{6}{.9} = \frac{60}{9} = \frac{20}{3}$

4. (PracticeExam #5)
Let $p(x) = x^3 - \frac{21}{8}x^2 + \frac{21}{4}x - \frac{5}{2}$
whose graph is shown to the right.



(a) (Quiz2 #2) [3] Will Newton's method always be able to find a root of this polynomial p no mater the starting value? Justify your answer.

Yes? Notice $p(x)$ will have a real root so Newton's method done just with $\mathbb{R}$ may lead to it.

sense (+1)

explain a fail (+.5)

explain why it won't happen (+1)

explain 2nd failure type (+.5)

Notice $p(x)$'s derivative never equals zero so our tangent lines will always cross the x-axis leading to the next term being defined.

Notice also $p(x)$'s concavity is such that we won't have any loops in Newton's method & will strictly approach our real root.

(b) (WebHW3 #1) [2] Choose an x_1 value, and identify x_2 using Newton's method (either numerically or graphically).

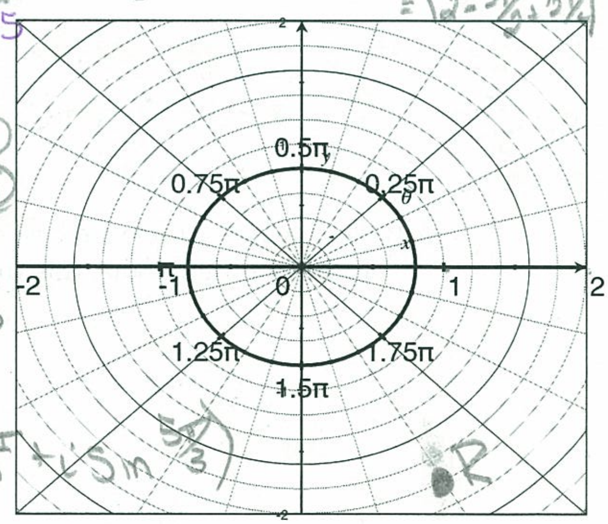
graphically shown above (id x_1 (+.5), ink tang line (+1), id x_2 (+.5))
or

$x_1 = 2$ $\quad p(2) = 8 - \frac{21}{2} + \frac{21}{2} - \frac{5}{2} = 11/2$ $\quad p'(2) = 3(4) - \frac{21}{4}(2) + \frac{21}{4}$
$= 12 - 21/2 + 21/4$

$x_2 = 2 - \frac{p(2)}{p'(2)} = 2 - \frac{11/2}{6.75} = 1.185$

(c) [1] The complex number $R = 1 - (\sqrt{3})i$ is a root of p. Plot R on the complex axis.

angle (+.5) radius (+.5) or $\mathbb{R}$ coord (+.5) i coord (+.5)



(d) [2] (ApxH #26) Convert R into polar coordinates.

(+.5) $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\tan ? = \frac{-\sqrt{3}}{1} = -\frac{\pi}{3}$ (+.5)

polar form (+.5)

(+.5) $2e^{-\frac{\pi}{3}i}$ or $2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

(e) (PracticeExam #6) [2] Compute $R^3 - 8$.

$(2e^{-\frac{\pi}{3}i})^3 - 8 = 2^3(e^{-\frac{\pi}{3}\cdot 3i}) - 8 = 8e^{-\pi i} - 8$
$= 8(-1) - 8$
$= -8 - 8 = -16$

DMS law (+1)
algebra (+1)

5. Consider the function $f(x) = -3\cos(x^2)$

(a) [4] (§11.10 #39) Find a power series representation for f. Any power series will suffice but supply work so I can see which one you are working with.

Recall $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$ } (+1)

start (+.5)
plan (+.5)

So $-3\cos(x^2) = -3\left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots\right)$

(+1.5) so $x^2 = u$ $\quad = -3\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots\right)$

(+.5) $\quad = -3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

(b) [2] For what x values will the power series of x converge to $f(x)$?

The $\cos(u)$ converges to the Taylor series for all $\mathbb{R}$ } (+1)

So $-3\cos(x^2)$ converges to the above series for all $\mathbb{R}$ } (+1)

(c) [4] Find a *reasonable* bound for the error of the second Taylor polynomial approximation centered at 0 for $f(3.5)$. Make sure that you show enough work that I know why you choose the M that you did. } (+.5)

error for (+.5) Error bounded by $\frac{M}{3!}(x-0)^3$ or $\frac{M}{3!}(3.5-0)^3$ or $\frac{1000}{3!}(3.5)^3$ (+.5) (+.5) (+.5)

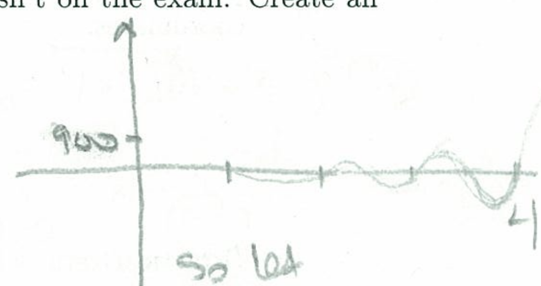
We need $M > f^{(3)}(t)$ for t between 0 and 3.5

derivatives (+1) $\quad f'(t) = +3(2t)\sin(t^2)$ $\qquad f'''(t) = -12t^2 \cdot 2t\sin(t^2) + 24t\cos(t^2)$

choose M (+.5) $\quad f''(t) = 6t \cdot 2t\cos(t^2) + 6\sin(t^2)$ $\qquad + 6 \cdot 2t\cos(t^2)$

$\qquad = -24t^3\sin(t^2) + 36t\cos(t^2)$

6 (d) [4] Identify a topic we covered in this class that wasn't on the exam. Create an exam question about this topic and answer it.



So let
$M = 1000$

start (+.5)
topic not on exam (+.5)
id'd topic (+1)
question (+1) (well defined +.5)
answer (+1)