note: complex numbers should have been in this too
#9 should be harder

Final

Tmath 126

Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges to a finite number.

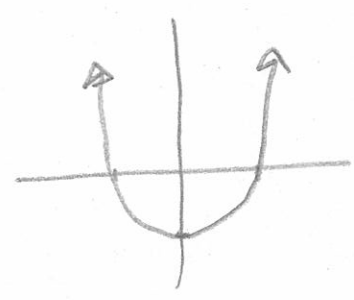
   false? the harmonic series $\sum_{n=1}^{\infty} 1/n$
   is such that $\lim_{n\to\infty} 1/n = 0$
   but the series does not converge.

   (b) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that the n^{th} partial sum of a series is $s_n = \dfrac{n+5n^2}{n^2-e}$. Then $\lim_{n\to\infty} a_n = 5$. false,

   Note $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ where s_n are the partial sums
   $= \lim_{n\to\infty} \dfrac{n+5n^2}{n^2-e} \overset{L'H}{=} \lim_{n\to\infty} \dfrac{10n}{2n} = 5$
   Because the series converges, the terms in the sequence must converge to zero.

   (c) If a function has a root, Newton's method will approach it no matter the initial guess.

   False ex $f(x) = x^2 - 1$

   if the first guess is 0,
   $f'(0) = 0$ and the expression $0 - \dfrac{f(0)}{f'(0)}$
   is not defined.
   It is equivalent to finding a line tangent to f that does not cross the x-axis

(d) $||\overrightarrow{a} \times \overrightarrow{b}|| = ||\overrightarrow{b} \times \overrightarrow{a}||$. True

$\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram with sides $\vec{a}$ & $\vec{b}$. Since this parallelogram has the same area as one with side lengths $\vec{b}$ & $\vec{a}$, the 2 expressions are equal.

(e) If $\overrightarrow{a} \cdot \overrightarrow{b} = 0$, then either $\overrightarrow{a} = 0$ or $\overrightarrow{b} = 0$ False

let $\vec{a} = \langle 0,1 \rangle$ and $\vec{b} = \langle 1,0 \rangle$

but $\vec{a} \cdot \vec{b} = 0$

(f) If $f(x, y)$ is a continuous function, the first-order derivatives exist, and f has a local minimum or maximum at the point $(0, 0)$, then $\nabla f(0, 0) = \overrightarrow{0}$.

True $\nabla f(0,0) = (f_x(0,0), f_y(0,0))$.

At a max/min the line tang. to f & ∥ to the x-axis is parallel to the xy plane $\Rightarrow$ $f_x(0,0) = 0$. Similarly for $f_y(0,0) = 0$.

(g) Let f be a function of x and y. If $\nabla f(c, d) = (2, 1)$, then the vector $\langle 2, 1 \rangle$ is tangent to the contour line of the surface of f at $(c, d, f(c, d))$.

False. $\nabla f(c,d)$ points in the direction of the steepest assent. The contour line would keep the 'elevation' constant.

(h) $\int_{-1}^{2} \int_{0}^{6} x^2 \sin(x - y)\, dx\, dy = \int_{0}^{6} \int_{-1}^{2} x^2 \sin(x - y)\, dy\, dx$

True the function $x^2 \sin(x-y)$ is con't on the rectangle $[0,6] \times [-1,2]$ so we can use Fubini's Thm.

(i) $\int_{-1}^{x} \int_{0}^{6} x^2 \sin(x - y)\, dx\, dy = \int_{0}^{6} \int_{-1}^{x} x^2 \sin(x - y)\, dy\, dx$

False the integral on the right is integrading over the shaded region

If we reversed the order of integration we'd see

$$\int_{-1}^{6} \int_{y}^{6} x^2 \sin(x-y)\, dx\, dy$$

2. Evaluate the following if possible.

$$\lim_{n\to\infty} \sin\left(\frac{6n\pi}{5+8n}\right)$$

$$= \sin\left(\lim_{n\to\infty} \frac{6n\pi}{5+8n}\right) \quad \text{b/c sin is con't}$$

$$= \sin\left(\lim_{n\to\infty} \frac{6n\pi\,(1/n)}{(5+8n)(1/n)}\right)$$

$$= \sin\left(\lim_{n\to\infty} \frac{6\pi}{\underbrace{5/n}_{\to 0}+8}\right)$$

$$= \sin\left(\lim_{n\to\infty} \frac{6\pi}{8}\right) = \sin\frac{3\pi}{4}$$

$$= \frac{1}{\sqrt{2}}$$

$$\sum_{n=0}^{\infty} \frac{n+1}{3n+2}$$

notice $\displaystyle\lim_{n\to\infty} \frac{n+1}{3n+2} = \lim_{n\to\infty} \frac{n+1}{3n+2}\frac{(1/n)}{(1/n)}$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}\to 0}{3+\frac{2}{n}\to 0} = 1/3 \neq 0$$

Thus the infinite sum will never converge to a finite number.

Diverges

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{1}{4} + \frac{-3}{4^2} + \frac{3^2}{4^3} + \ldots$$

Geometric series where $a = 1/4$ $r = -3/4$

Since $-1 < r < 1$, the series converges to

$$\frac{a}{1-r} = \frac{1/4}{1-{-3/4}} = \frac{1/4}{7/4} = 1/7$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!}$$

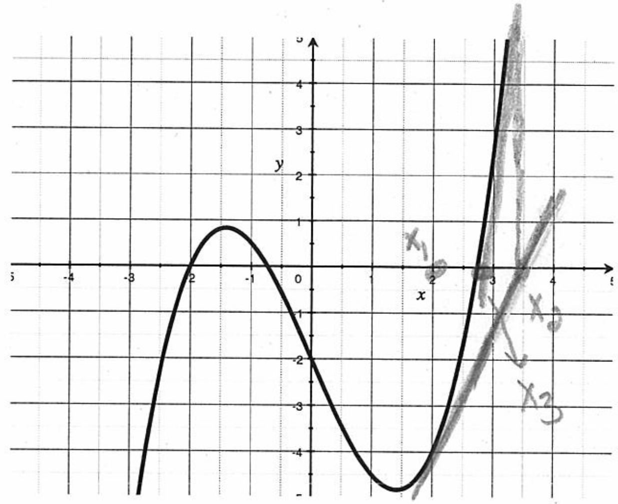
looks alot like the series for cosine: $\displaystyle \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

but instead of having a "x^{2n}" we have a 1

So

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{2n}}{(2n)!} = \cos 1$$

3. The graph of $g(x) = \frac{1}{2}x^3 - 3x - 2$ is shown to the right.



(a) Choose an initial value so that Newton's method will converge to the positive root. 2

Use the graph on the right to estimate the first three approximations in Newton's method.

(b) Find the second order Taylor polynomial $T_2(x)$ at $b = 2$.

K	$g^k(x)$	$g^k(2)$
0	$\frac{1}{2}x^3-3x-2$	-4
1	$\frac{3}{2}x^2-3$	3
2	$3x$	6

Recall $T_2(x)$ centered at b is:

$$g(2) + \frac{g'(2)}{1!}(x-b) + \frac{g''(2)}{2!}(x-b)^2$$

$$\Rightarrow T_2(x) = -4 + 3(x-2) + \frac{6}{2}(x-2)^2$$

$$= -4 + 3(x-2) + 3(x-2)^2$$

(c) Approximate $g(2.2)$ using $T_2(x)$.

$$g(2.2) \approx T_2(2.2) = -4 + 3(2.2-2) + 3(2.2-2)^2$$

$$= -4 + 3 \cdot .2 + 3 \cdot .2^2 = -3.28$$

(d) Use Taylor's inequality to find an upper bound for the error in the approximation above.

Recall that $|\text{error}| \leq \frac{M}{3!}|x-2|^3$ where M is such that $|g^{(3)}(x)| \leq M$ for all $x \ni |x-2| \leq .2$

note $g^{(3)}(x) = 3$ so we can let $M=3$

So the $|\text{error}| < \frac{3}{3!}|2.2-2|^3 = \frac{1}{2} \cdot .2^3$

$\Rightarrow |\text{error}| < .004$

note $f(2.2) = -3.276$

$T_2(2.2) = -3.28$

$.004$

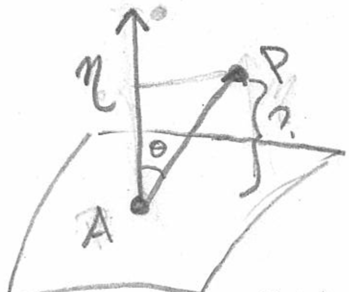
So we actually hit this (maximal error.

3. (a) (5 points) Find the distance between the plane $x-y+2z=3$ and the point (2,-1,3). "P"

notice that the vector $\underbrace{\langle 1,-1,2\rangle}$ is normal to the given plane. $\left(1(x-0)+-1(y+1)+2(z-1)=0\right)$

and the point $A=(0,-1,+1)$ is on the plane

We can use properties of the dot product to figure out θ + then use Sohcahtoa to find the length of ?. $\overrightarrow{AP}=\langle 2-0,-1+1,3-1\rangle=\underbrace{\langle 2,0,2\rangle}$



note $\|\vec{\eta}\|\|\overrightarrow{AP}\|\cos\theta=\vec{\eta}\cdot\overrightarrow{AP}$

$\Rightarrow \cos\theta=\dfrac{2+0+4}{\sqrt{1+1+4}\sqrt{4+4}}=\dfrac{6}{\sqrt{6}\sqrt{8}}$

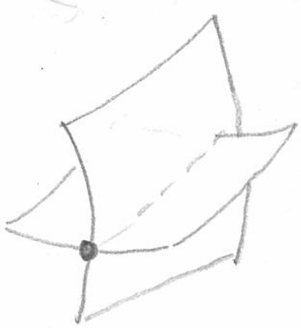
Sohcahtoa $\Rightarrow \cos\theta=\dfrac{\|?\|}{\|\overrightarrow{AP}\|}$

$\dfrac{6}{4\sqrt{3}}=\dfrac{\|?\|}{\sqrt{4+4}}$

$\Rightarrow \|?\|=\dfrac{6}{4\sqrt{3}}\cdot 2\sqrt{2}=\dfrac{3\sqrt{2}}{\sqrt{3}}$

$=\sqrt{6}$

(b) (5 points) Find the equation of the line of intersection between $x-y+2z=3$ and $x+2y+3z=0$.



Denote $x-y+2z=3$ as plane P and

$x+2y+3z=0$ as plane Q.

Now plane P intersects the xy plane at $x-y=3$

and plane Q intersects the xy plane at $x+2y=0$

So the point in the xy plane that both P & Q share

is $\left\{\begin{matrix} x-y=3 \\ x+2y=0\end{matrix}\right\} \xrightarrow{x=3+y} (3+y)+2y=0 \Rightarrow 3+3y=0 \Rightarrow y=-1,\ x=2$

(2,-1,0)

Since the line is in both plane P & plane Q, the line must be ⊥ to both ⟨1,-1,2⟩ and ⟨1,2,3⟩ (their respective normal vectors. Thus we can find the directional vector for the line with the cross product

$\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & 2 & 3\end{vmatrix}=i(-3-4)-j(3-2)+k(2+1)\Rightarrow\langle -7,-1,3\rangle$

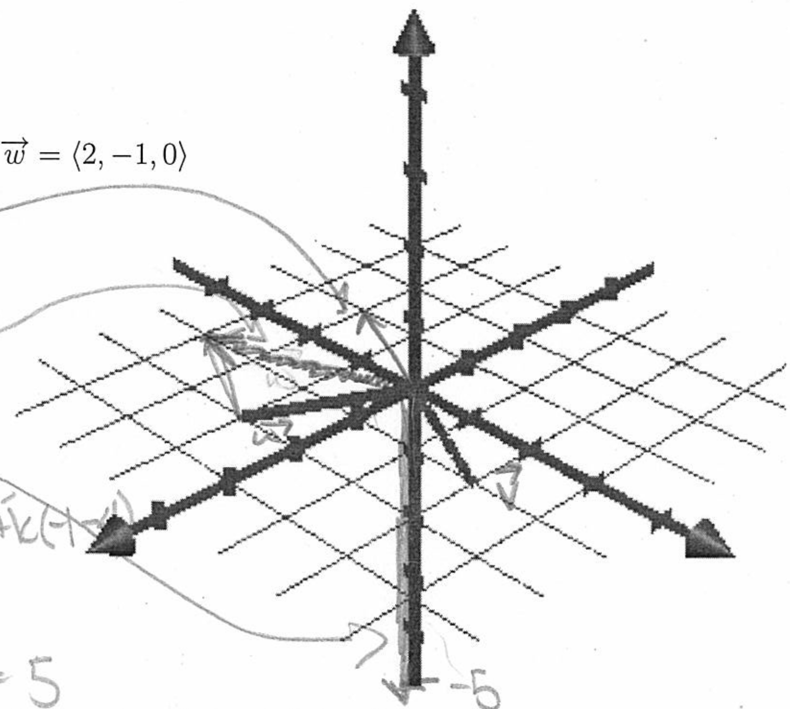
So

$(2,-1,0)+t\langle -7,-1,3\rangle$

as $t\in\mathbb{R}$ works.

Consider the vectors: $\vec{v} = \langle 1, 2, 0 \rangle$ and $\vec{w} = \langle 2, -1, 0 \rangle$

(a) [1] Draw the vector $-\vec{v}$



(b) [2] Draw the vector $\vec{w} - \vec{v}$

(c) [2] Draw the vector $\vec{v} \times \vec{w}$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 2 & -1 & 0 \end{vmatrix} = i(0-0) - j(0-0) + k(-1-4)$$

$$= -5k$$

$$\Rightarrow \|\vec{v} \times \vec{w}\| = 5$$

6. Consider the quadratic surface given by the equation $2x^2 + 3y^2 - 5z^2 = 0$.

(a) Describe the cross sections created when sliced parallel to the xy plane. How about for those parallel to the yz plane.

if $z=0 \Rightarrow 2x^2+3y^2=0$ dot

$z=1 \Rightarrow 2x^2+3y^2=5$ ellipse

in general when z is fixed the cross sections (sliced // to xy plane) will be ellipses.

if $x=0 \Rightarrow 3y^2-5z^2=0$ dot

$x=1 \Rightarrow 3y^2-5z^2=-2$ hyperbola

in general the cross sections // to the yz plane are hyperbolas.

(b) Find the equation of the tangent plane to the surface at the point $(1, 1, 1)$.

I'll use $z-z_1 = m_x(x-x_1)+m_y(y-y_1)$ since I can write

$$f(x,y) = \begin{cases} \sqrt{\frac{1}{5}(2x^2+3y^2)} & \text{if } z>0 \\ -\sqrt{\frac{1}{5}(2x^2+3y^2)} & \text{if } z<0 \end{cases}$$

But we are considering $(1,1,1)$ so we can isolate the top eq.

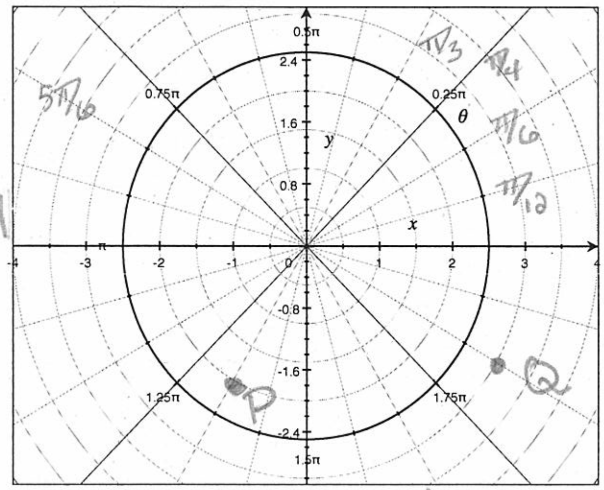
$$f_x(x,y) = \frac{1}{2}\left(\frac{1}{5}(2x^2+3y^2)\right)^{-1/2} \cdot \frac{4}{5}x = \frac{2}{5\sqrt{5}}x(2x^2+3y^2)^{-1/2} \Rightarrow f_x(1,1) = \frac{2 \cdot 1}{5\sqrt{5}} 5^{-1/2} =$$

$$f_y(x,y) = \frac{1}{2}\left(\frac{1}{5}(2x^2+3y^2)\right)^{-1/2} \cdot \frac{6}{5}y = \frac{3}{5\sqrt{5}}(2x^2+3y^2)^{-1/2} \Rightarrow f_y(1,1) = \frac{3}{5\sqrt{5}} 5^{-1/2} = \frac{3}{25}$$

So $z-1 = \frac{2}{25}(x-1) + \frac{3}{25}(y-1)$

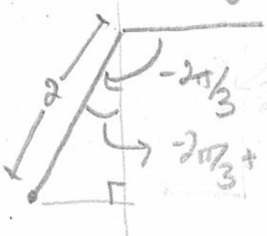
7. Let $P(2, -\frac{2\pi}{3})$ and $Q(-3, \frac{5\pi}{6})$ be polar coordinates.

(a) [2] Plot P and Q.



Sohcahtoa

(b) [2] Find the Cartesian coordinates of the point P.



$\hookrightarrow -2\pi/3 + \pi/2 = -\pi/6$

$\sin -\pi/6 = \frac{x}{2}$

$\Rightarrow x = 2\sin -\pi/6 = 2 \cdot -\frac{1}{2} = -1$

$\cos -\pi/6 = y/2$

$\Rightarrow y = 2\cos -\pi/6 = 2 \cdot \sqrt{3}/2 = \sqrt{3}$

So $(-1, \sqrt{3})$

(c) [2] Find two other pairs of polar coordinates for the point P. $(2, -2\pi/3 + 2\pi)$ $(2, -2\pi/3 + 4\pi)$ $(-2, \pi/3)$ etc

8. Consider the function $h(x, y) = x^3 - 12xy + 8y^3$.

(a) Find all critical points of h.

Critical points are when $h_x(x,y) = h_y(x,y) = 0$

$h_x(x,y) = 3x^2 - 12y \Rightarrow 0 = 3x^2 - 12y \quad y = \frac{3x^2}{12} = x^2/4$

$h_y(x,y) = 24y^2 - 12x \quad 0 = 24y^2 - 12x$

$\Rightarrow 0 = 24\left(\frac{x^2}{4}\right)^2 - 12x$

$\hookrightarrow 0 = \frac{2 \cdot 4 \cdot 3}{4 \cdot 4}x^4 - 12x = \frac{3}{2}x^4 - 12x$

$0 = \frac{1}{2}x(3x^3 - 24) \Rightarrow \frac{1}{2}x = 0$ or $3x^3 - 24 = 0$

$\Rightarrow x = 0 \quad x^3 = \frac{24}{3} = 8 \Rightarrow x = 0$ or $x = 2$

$\rightarrow$ when $x = 0$ $y = 0 \Rightarrow (0,0)$

when $x = 2$, $y = 1 \Rightarrow (2,1)$

(b) Classify each critical point as a local minimum, a local maximum, or a saddle point.

Since I wasn't given a picture, I'll have to use the 2nd der test.

$h_{xx}(x,y) = 6x$

$h_{yy}(x,y) = 48y$

$h_{xy}(x,y) = -12$

(0,0):

$f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2$

$= 0 - (-12)^2 = -144 < 0$

not a local min or max

(2,1):

$f_{xx}(2,1)f_{yy}(2,1) - [f_{xy}(2,1)]^2$

$12 \cdot 48 - (-12)^2 > 0$

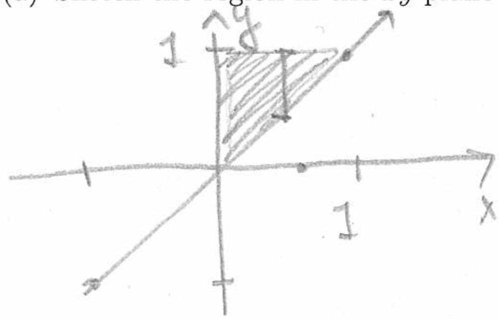
and $f_{xx}(2,1) = 12 > 0$

local minimum

9. Consider the double integral

$$\int_0^1 \int_x^1 e^{\frac{x}{y}}\, dydx$$

(a) Sketch the region in the xy-plane where the integral is taken over.



$\int dx \times \int dy$ 1st

(b) Switch the order of integration.

$$\int_0^1 \int_0^y e^{x/y}\, dx\, dy$$

(c) Compute the double integral.

$$\int_0^1 \left(\int_0^y e^{x/y}\, dx \right) dy = \int_0^1 y e^{x/y} \Big]_0^{x=y} dy$$

b/c $\frac{d}{dx}\left(y e^{x/y}\right) = y e^{x/y} \cdot \frac{1}{y}$

$$= \int_0^1 y e^{y/y} - y e^{0/y}\, dy = \int_0^1 y e^1 - y \cdot 1\, dy$$

$$= \int_0^1 ey - y\, dy = \frac{e}{2} y^2 - \frac{1}{2} y^2 \Big]_0^1$$

b/c $\frac{d}{dy}\left(\frac{e}{2} y^2 - \frac{1}{2} y^2\right) = \frac{e}{\cancel{2}} y \cdot \cancel{2} - \frac{1}{\cancel{2}} y \cancel{2}$

$$= \left(\frac{e}{2} 1^2 - \frac{1}{2} 1^2\right) - \left(\frac{e}{2} \cdot 0^2 - \frac{1}{2} \cdot 0^2\right)$$

$$= \frac{e}{2} - \frac{1}{2} = \frac{1}{2}(e-1)$$