note: I didn't have a calculator
so I left answers in exact form

Key

Final **Tmath 126** Summer 2011

1. [12] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

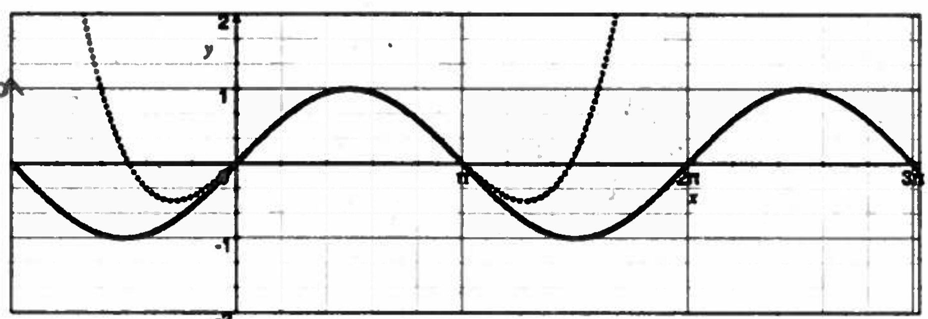
Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

(a) A series is an "infinite sum" and never converges to a finite number.

False, consider the "infinite sum" $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$
This geometric series converges to 1

(b) The dotted function below is the 4th Taylor polynomial of $\sin(x)$ centered at 0.

False, ~~reason~~
The dotted function does look like a 4th deg poly, but it looks centered at $\pi/2$ not 0.



(c) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

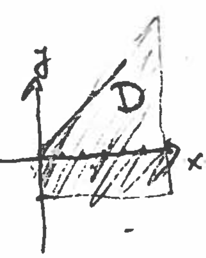
True. The vector $(\vec{a}\times\vec{b})$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
Thus the angle between $\vec{a}$ & $\vec{a}\times\vec{b}$ is 90°.
$(\vec{a}\times\vec{b})\cdot\vec{a} = \|\vec{a}\times\vec{b}\|\cdot\|\vec{a}\|\cos 90° = 0$
"def of dot product"

(d) $\int_{-1}^{x}\int_0^6 x^2\sin(x-y)\,dx\,dy = \int_0^6\int_{-1}^{x} x^2\sin(x-y)\,dy\,dx$

False

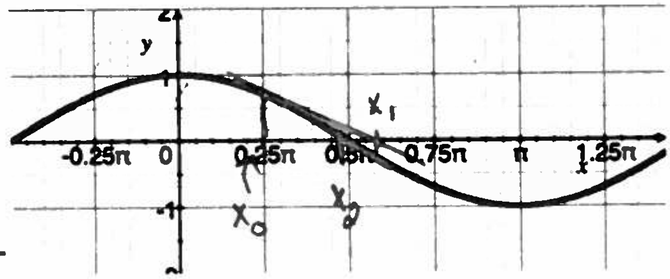
returns a function of x

returns volume (signed) trapped above/below the region D under the surface $z = x^2\sin(x-y)$



Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Recall: $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$



(a) [1] Choose an initial value so that Newton's method will not find a root of $\cos(x)$. $0, \pi$ both work

(b) [2] Choose an initial value so that Newton's method does converge to a root and identify the first three approximations on the graph to the right. identified x_0, x_1 & x_2 (and tangent lines used to find x_i)

(c) [2] Find whether the *sequence* $a_n = (-1)^n \frac{(\frac{\pi}{4})^{2n}}{(2n)!}$ converges or diverges. If it converges, find the limit. *Justify* your answer.

From above/Taylor series we know $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ is true for all x. So $\sum_{n=0}^{\infty}(-1)^n \frac{(\pi/4)^{2n}}{(2n)!} = \cos \pi/4 = 1/\sqrt{2}$.

We were asked about the sequence a_n. Recall if $\sum a_n$ converges $\lim_{n\to\infty} a_n = 0$. Thus $\lim_{n\to\infty} a_n = 0$.

(d) [5] Find the second order Taylor approximation of $5\cos x$ centered at π.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$5\cos x$	-5
1	$-5\sin x$	0
2	$-5\cos x$	5

Recall $T_2(x) = f(b)(x-b)^0 + \frac{f'(b)}{1!}(x-b)^1 + \frac{f''(b)}{2!}(x-b)^2$

So $-5(x-\pi)^0 + \cancel{\frac{0}{1!}(x-\pi)} + \frac{5}{2!}(x-\pi)^2$

$-5 + \frac{5}{2}(x-\pi)^2$

(e) [1] Approximate $5\cos 3$ using your above Taylor approximation.

Notice $\pi \approx 3$ so $5\cos 3 \approx -5 + 5/2(3-\pi)^2$

(f) [4] Use Taylor's inequality to find an upper bound for the error in the approximation above.

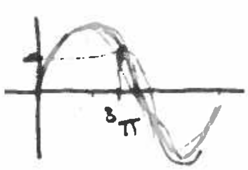
Recall error bound for T_2 is $\frac{M}{3!}(x-b)^3$

where x-b is the worse case (here it will be $\pi - 3$)

and M is such that $|M| \geq |f^{(3)}(x)|$ for all x between π & 3

$f^{(3)}(x) = 5\sin x$

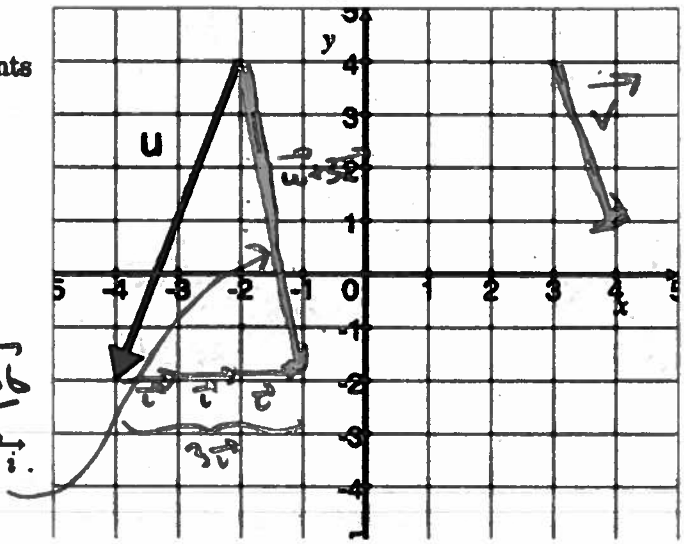
we can let M be $5\sin 3$ (or slightly larger - 5 would work but it'd be silly)



So error is bounded by $\left|\frac{\sin 3}{3!}(3-\pi)^3\right| = \frac{(\sin 3)(\pi-3)^3}{6}$

$\approx$

3. The vector $\vec{u} \in \mathbb{R}^2$ and shown below, answer the following:



(a) [1] What are the components of $\vec{u}$?

$\langle -2, -6 \rangle$

(b) [1] Draw the vector $\vec{v} = \langle 1, -3 \rangle$.

(c) [1] Find $\|\vec{u}\|$.

$\sqrt{(-2)^2 + (-6)^2} = \sqrt{4+36}$
$= \sqrt{40}$

(d) [2] Draw the vector $\vec{u} + 3\,\vec{i}$.

(e) [2] Find the angle between $\vec{u}$ and $\vec{v}$.

Recall $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$

$\Rightarrow (-2)(1) + (-6)(-3) = \sqrt{40}\sqrt{1^2 + (-3)^2} \cos\theta$

$16 = \sqrt{40}\sqrt{10} \cos\theta$

$\frac{16}{\sqrt{400}} = \cos\theta$

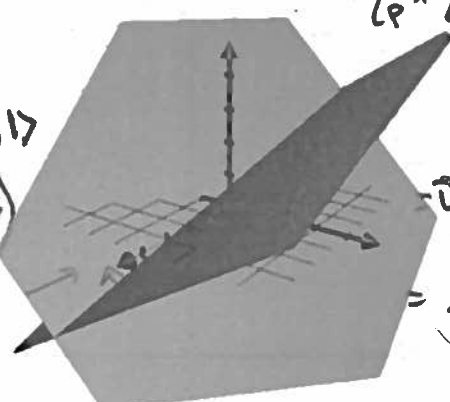
$\frac{16}{10 \cdot 2} = \cos\theta$

$8/10 = \frac{4}{5} = \cos\theta$

$\theta = \cos^{-1}(4/5)$

4. Consider the planes P and Q defined by $x + y + z = -1$ and $x - 2y + 2z = 1$ respectively and drawn below.

[4] Notice that the two planes intersect and form a line. Find an equation of the line of intersection.



P : $x+y+z = -1$ has the normal $\eta_P = \langle 1,1,1 \rangle$

$x+y+z+1 = 0 \Rightarrow \langle 1,1,1 \rangle \cdot (x-0, y-0, z+1)$

or $\langle 1,1,1 \rangle \cdot (\langle x, y, z \rangle - \langle 0,0,-1 \rangle)$

and passes thru $(0,0,-1)$

Q : $x - 2y + 2z = 1$ similarly found has $\eta_Q = \langle 1, -2, 2 \rangle$ and passes thru $(1,0,0)$

$\eta_P \times \eta_Q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{vmatrix}$

$= \vec{i}(2+2) - \vec{j}(2-1) + \vec{k}(-2-1)$

$= 4\vec{i} - \vec{j} - 3\vec{k}$

1) Finding a point on the line of intersection: I'll set $z=0$ to find where the line crosses the xy plane

$\begin{cases} x+y+0 = -1 \\ x-2y+2\cdot 0 = 1 \end{cases} \Rightarrow x = -1-y$ so $(-1-y) - 2y = 1 \Rightarrow -3y = 2$

$y = -2/3$, $x = -1/3$, $z = 0$

2) The directional vector is $\perp$ to η_P and η_Q so we need the cross product.

$\langle 4, -1, -3 \rangle$

3) so

$\langle -1/3, -2/3, 0 \rangle + t\langle 4, -1, -3 \rangle$

as t varies gives the equation of a line.

5. [3] Match the following equations to their respective graphs:

A. $z = \cos x$

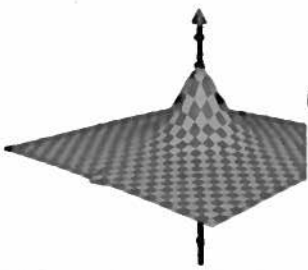
notice y does not appear => 'cylinder'

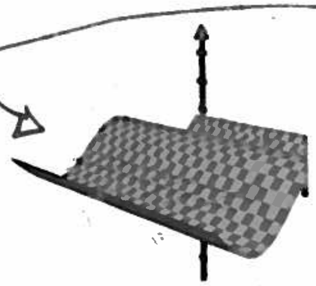
B. $x^2 - y^2 + z^2 = 1$

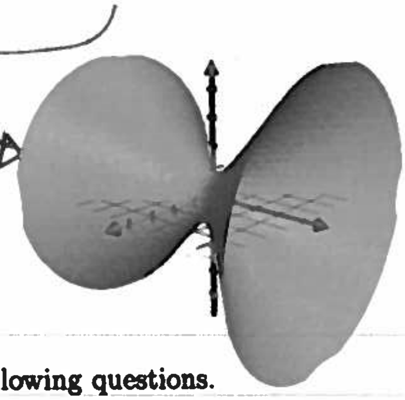
notice if y is 0 or a constant the contours are circles

C. $z = \dfrac{3}{1 + x^2 + y^2}$ $\Big\}$ as x and y → ∞ z → 0 ok

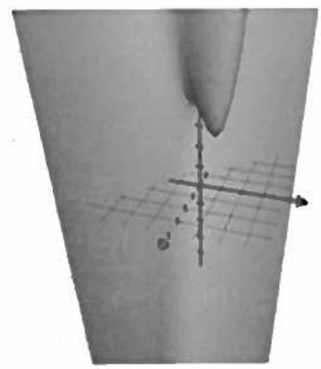
$\frac{3}{\text{Big}}$ = little







6. Consider the function $f(x, y) = 4 + x^3 + y^3 - 3xy$ for the following questions. The graph and contour lines are provided below.



(a) [3] Find the gradient of f.

$\nabla f = \langle f_x, f_y \rangle$

$= \langle 3x^2 - 3y, 3y^2 - 3x \rangle$

(b) [3] Find the equation of the plane tangent to the surface when $x = 2$ and $y = 1$.

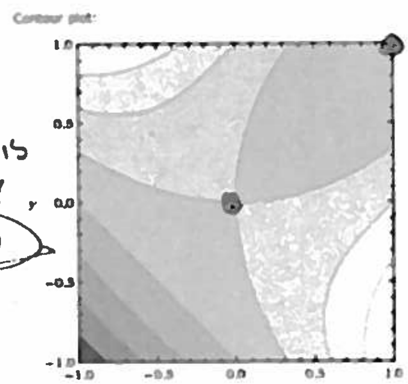
I'll use: $z - z_1 = m_x(x - x_1) + m_y(y - y_1)$

$m_x = f_x(2,1) = 3(2)^2 - 3(1) = 9$

$m_y = f_y(2,1) = 3(1)^2 - 3(2) = -3$

if $x = 2$ & $y = 1$ then $f(2,1)$ is $4 + 8 + 1 - 6 = 7$,

$z - 7 = 9(x - 2) - 3(y - 1)$



(c) [4] Find all critical points and classify them as local minimums, maximums, or saddle points.

Critical Points happen when $f_x(x,y) = f_y(x,y) = 0$ or $f_x(x,y)$ & $f_y(x,y)$ DNE.

Notice the domains of f_x + f_y are $\mathbb{R}$ so we need only find the zeros.

$$\begin{Bmatrix} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x^2 = y \\ y^2 = x \end{Bmatrix} \Rightarrow \begin{matrix} x = 0 \;\&\; y = 0 \\ x = 1 \;\&\; y = 1 \end{matrix}$$

Considering the contour lines (1,1) is a local min & (0,0) is a saddle.

7. [4] Find $\int_0^1 \int_0^2 xye^{x^2y}\,dy\,dx$.

$$= \int_0^2 \left(\int_0^1 xye^{x^2y}\,dx \right) dy \quad \text{by Fubini}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2y} \right]_{x=0}^{1} dy \quad \text{by} \rightarrow$$

Consider $\int xye^{x^2y}\,dx$ (or $\int 7xe^{7x^2}\,dx$)

let $u = x^2y$

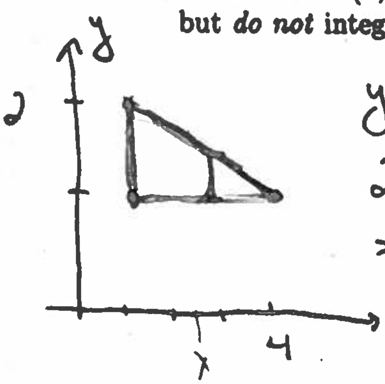
$\Rightarrow du = 2xy\,dx$ or $\frac{1}{2}du = xy\,dx$

$\int xye^{x^2y}\,dx = \int \frac{1}{2} e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{x^2y}$

$$= \int_0^2 \frac{1}{2}e^y - \frac{1}{2}e^0\,dy = \int_0^2 \frac{1}{2}e^y - \frac{1}{2}\,dy = \left[\frac{1}{2}e^y - \frac{1}{2}y\right]_0^2$$

$$= \left(\frac{1}{2}e^2 - \frac{1}{2}\cdot 2\right) - \left(\frac{1}{2}e^0 - \frac{1}{2}\cdot 0\right) = \frac{1}{2}e^2 - 1 - \frac{1}{2} = \frac{1}{2}e^2 - \frac{3}{2}$$

8. [3] Consider the volume under the surface of the curve $z = xy$ and above the triangle with vertices $(1,1)$, $(4,1)$, and $(1,2)$. Set up the definite integral to find this volume but *do not* integrate.



$y = \frac{-1}{3}x + b$

$2 = -\frac{1}{3}(1) + b$

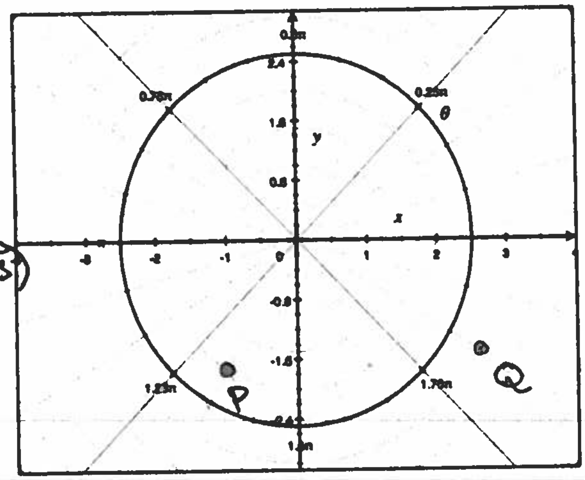
$\Rightarrow b = 2 + \frac{1}{3} = \frac{7}{3}$

$y = \left(\frac{-1}{3}\right)x + \frac{7}{3}$

$$\int_1^4 \int_1^{(-\frac{1}{3})x + \frac{7}{3}} xy\,dy\,dx$$

9. Let $P(2, -\frac{2\pi}{3})$ and $Q(-3, \frac{5\pi}{6})$ be polar coordinates.

(a) [2] Plot P and Q.



(b) [2] Find the Cartesian coordinates of the point P.

$2e^{-i\frac{2\pi}{3}} = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

$= 2\left(-\frac{1}{2} + i\frac{-\sqrt{3}}{2}\right) = -1 - i\sqrt{3}$ or $(-1, -\sqrt{3})$

or you can use Sohcahtoa...

(c) [2] Find two other pairs of polar coordinates for the point P.

$2e^{i\frac{4\pi}{3}}$ or $2e^{i\frac{10\pi}{3}}$ etc

$(2, \frac{4\pi}{3})$ $\quad$ $(2, \frac{10\pi}{3})$

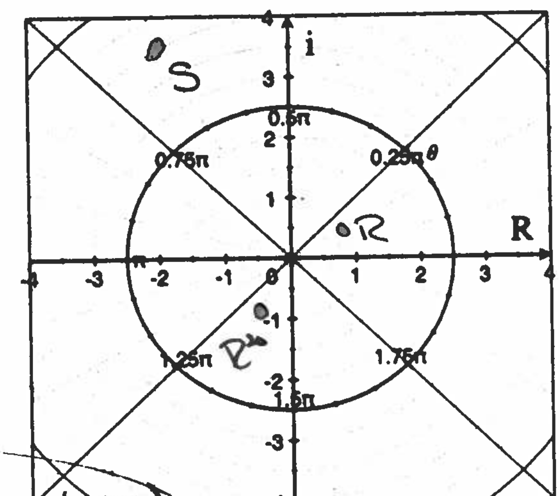
10. Let R and S be complex numbers where $R = e^{\frac{i\pi}{6}}$ and $S = -2 + (2\sqrt{3})i$

(a) [1] Convert R to rectangular coordinates (that is, of the form $a + bi$).

I'll use Sohcahtoa this time

$\cos\frac{\pi}{6} = \frac{a}{1} \Rightarrow a = \frac{\sqrt{3}}{2}$ $\quad$ so

$\sin\frac{\pi}{6} = \frac{b}{1} \Rightarrow b = \frac{1}{2}$ $\quad$ $\frac{\sqrt{3}}{2} + i\frac{1}{2}$



(b) [1] Convert S to polar form (that is, of the form $re^{i\theta}$).

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 4 \cdot 3} = \sqrt{16} = 4$

$\tan\varphi = \frac{2\sqrt{3}}{2} = \sqrt{3}$ $\quad$ so $\theta = \frac{2\pi}{3}$

$\Rightarrow \varphi = \frac{\pi}{3}$ $\quad$ $4e^{i\frac{2\pi}{3}}$ or $(4, \frac{2\pi}{3})$

φ θ -2

(c) [1] Plot R and S on the complex plane provided to the right.

(d) [1] Multiply $R \cdot S$. (Note: this question didn't make sense in question 8!)

$R \cdot S = e^{i\frac{\pi}{6}} \cdot 4e^{i\frac{2\pi}{3}} = 4e^{i(\frac{\pi}{6} + \frac{2\pi}{3})} = 4e^{i\frac{5\pi}{6}}$

or $\left(\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right)\left(-2 + 2\sqrt{3}i\right)$ and then foil...

(e) [2] Multiply R^{20} and then plot this point on the axes provided on the right.

$(x^3)^2 = x^3 x^3$
$= xxx\,xxx$
$= x^6$

$R^{20} = \left(e^{i\frac{\pi}{6}}\right)^{20} = e^{i\frac{\pi}{6} \cdot 20} = e^{i\frac{10\pi}{3}}$ or $e^{i\frac{4\pi}{3}}$