Key

8:32
7:57
=> 35 min

Exam 2 — **Tmath 126** — Summer 2011

1. [12] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   start (+.5)
   T/F correct (+.5)
   looking for approp (+.5)
   found one (+1)
   sense/notation (+.5)

   (a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. True

   Geometric argument: because $\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\theta = \|\vec{b}\|\cdot\|\vec{a}\|\cos\theta = \vec{b}\cdot\vec{a}$
   and the angle between $\vec{a}$ & $\vec{b}$ is the same as that between $\vec{b}$ & $\vec{a}$.

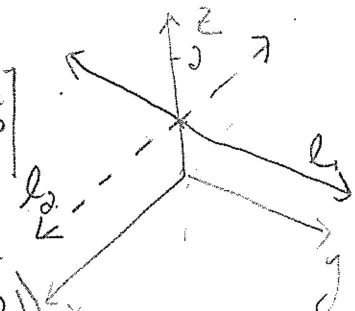
   Algebraic argument: Let $\vec{a} = \langle a_1, a_2, a_3\rangle$ & $\vec{b} = \langle b_1, b_2, b_3\rangle$.
   Then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\cdot\vec{a}$
   by commutivity of $\mathbb{R}$

   (b) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

   True $\vec{a}\times\vec{b}$ is a vector
   perpendicular to both $\vec{b}$ and $\vec{a}$.
   Thus the angle between $\vec{a}\times\vec{b}$ and $\vec{a}$ is $\frac{\pi}{2}$
   $\Rightarrow 0 = \cos(\frac{\pi}{2})\|\vec{a}\|\|\vec{a}\times\vec{b}\| = (\vec{a}\times\vec{b})\cdot\vec{a}$

   (c) Two lines parallel to a plane are parallel.

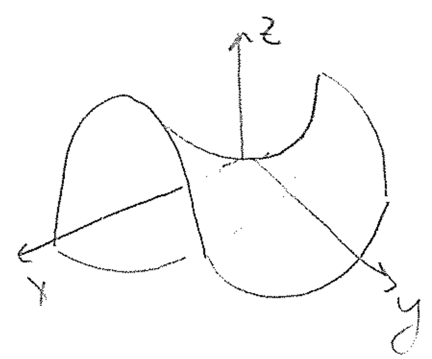
   False the lines: $l_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\t\\1\end{bmatrix}$ & $l_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}t\\0\\1\end{bmatrix}$
   where $t \in \mathbb{R}$

   are both // to the xyplane
   but the intersect at $(0,0,1)$

   (d) If $f(x, y)$ is a continuous function, the first-order derivatives exist, and $f_x(0,0) = 0 = f_y(0,0)$, then f has a local minimum or maximum at the point $(0,0)$.

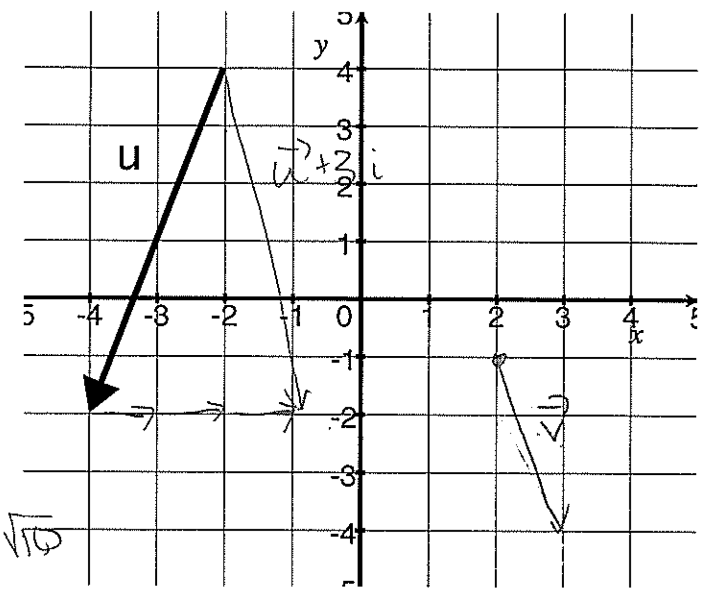
   false - there may be a
   saddle point which
   is neither a local
   max nor local min

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

typo

2. The vector~~s~~ $\vec{u} \in \mathbb{R}^2$ and shown below, answer the following:



(a) [1] What are the components of $\vec{u}$?

$\langle -2 , -6 \rangle$

+.5 +.5

typo

(b) [1] Draw the vector $\vec{v} =$ ~~$\langle 1, 4 \rangle$~~ $\langle 1, -3 \rangle$

(c) [1] Find $\|\vec{u}\|$.

$\sqrt{(-2)^2 + (-6)^2} = \sqrt{4+36}$

$= \sqrt{40} = 2\sqrt{10}$

(d) [2] Draw the vector $\vec{u} + 3\vec{i}$.

+.5 +.5

add vectors +.5

got it +.5

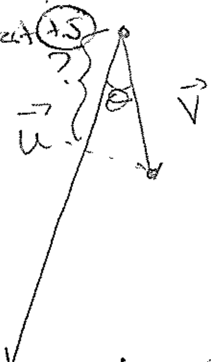
(e) [2] Find the angle between $\vec{u}$ and $\vec{v}$. Recall $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$ +.5

$\to \theta = \cos^{-1}\left(\dfrac{\langle -2,-6\rangle \cdot \langle 1,-3\rangle}{2\sqrt{10}\,\sqrt{1^2+(-3)^2}}\right) = \cos^{-1}\left(\dfrac{-2+18}{2\sqrt{10}\sqrt{10}}\right) = \cos^{-1}\left(\dfrac{16}{20}\right) = \cos^{-1}\left(\dfrac{4}{5}\right)$

+.5 +.5 got it +.5

(f) [4] Find the projection of $\vec{v}$ onto $\vec{u}$.

Start +.5



Sohcahtoa

$\Rightarrow \cos\theta = \dfrac{\|?\|}{\|\vec{v}\|}$

$\Rightarrow \|?\| = \|\vec{v}\|\cos\theta$

$= \sqrt{10}\cos\theta$ (!!) +.5

dot product def

$\Rightarrow \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$ +.5

$= \dfrac{4}{5}$ from (e) (*)

+1 perfect if you've formula written/used wrong

Use (*) to sub into (!!) and we know

$\|?\| = \sqrt{10} \cdot \dfrac{4}{5}$ now we only need direction... +.5

$\|?\| \cdot \dfrac{\vec{u}}{\|u\|} = \dfrac{4\sqrt{10}}{5} \cdot \dfrac{1}{2\sqrt{10}} \langle -2,-6\rangle = \dfrac{2}{5}\langle -2,-6\rangle$

+.5 mag. +1 dir/unit

$= \left\langle -\dfrac{4}{5}, -\dfrac{12}{5} \right\rangle$ alg/notation +.5

3. [3] Find the equation of the plane that passes through the points $\overset{P}{(0,1,1)}$, $\overset{Q}{(1,0,1)}$, and $\underset{R}{(1,1,0)}$.

eq of line (.5) notation/sense (.5)

Now $\vec{PQ} = \langle 1,-1,0\rangle$ and $\vec{PR} = \langle 1,0,-1\rangle$,

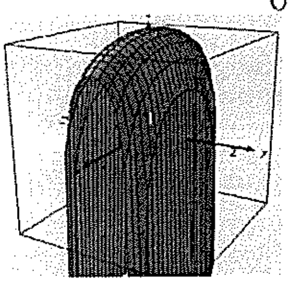
(+1) { $\vec{PQ} \times \vec{PR}$ will give us a normal line $\begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i}(1-0) - \vec{j}(-1-0) + \vec{k}(0+1)$

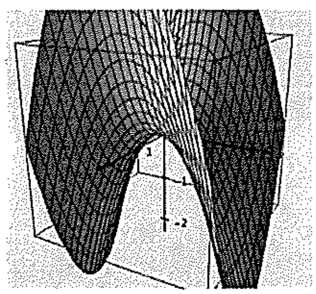
So $\vec{n} = \vec{i} + \vec{j} + \vec{k}$.

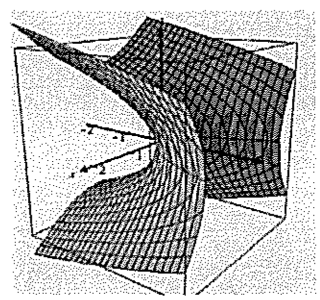
used eq's correctly (+1) ✓ $\Rightarrow \langle 1,1,1\rangle \cdot (\langle x,y,z\rangle - \langle 0,1,1\rangle) = 0$ works.

or $\langle 1,1,1\rangle \cdot (\langle x, y-1, z-1\rangle) = 0$

or $x + y - 1 + z - 1 = 0$ or $x + y + z = 2$

i) 

ii) 

iii) 

4. [3] Consider the three graphs above for the following questions:

typo

(a) Match the following equations to their respective graphs:

A. $z = x^2 - y^2$ — ii

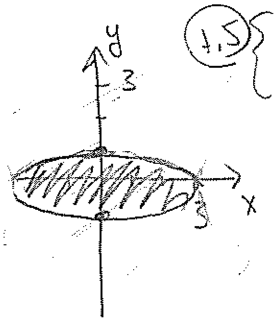
B. $y = x^2 - z^2$ — iii

C. $z = \ln(9 - x^2 - 9y^2)$ — i)

(b) [1] Identify which of the above are graphs of functions.

$\underbrace{\text{i}}_{(.5)}$ & $\underbrace{\text{ii}}_{(.5)}$

(c) [2] For each of the expressions above that are functions, identify the domain and the range.

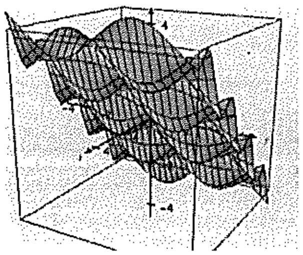
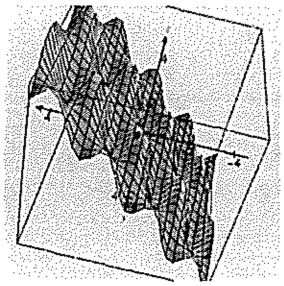
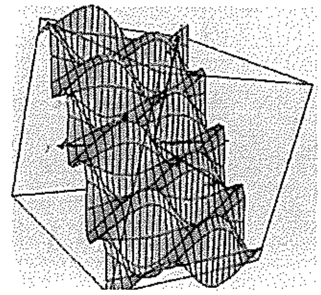


(.5){ ii Domain : $\mathbb{R}^2$ Range: $\mathbb{R}^2$

i (+1){ Domain $\{(x,y) \mid 9 - x^2 - 9y^2 > 0\} = \{(x,y) \mid 9 > x^2 + 9y^2\}$ so a region bounded by a ellipse

(.5){ Range: from the graph the max is at $(0,0)$ so $(-\infty, \ln(9)]$

5. Three views of the function $f(x, y) = x + \cos(3x)\sin(y)$ are shown below and may be used for the following questions. The point $(0, \frac{\pi}{2}, f\left(0, \frac{\pi}{2}\right))$ is identified on the graph.

y is const (+.5)
chain rule (+.5)

(a) [3] Find the gradient of f.

$\nabla f = \langle f_x, f_y \rangle$

$f_x(x,y) = 1 - \sin(y)\sin(3x)\cdot 3$ (+.5)

$f_y(x,y) = \cos(3x)\cos(y)$ } (+1)

$= \langle 1 - 3\sin(3x)\sin(y),\ \cos(3x)\cos(y) \rangle$

notation (+.5)

(b) [1] Evaluate the gradient at the point $(0, \frac{\pi}{2})$. (+.5) eval everywhere (+.5)

$= \langle 1 - 3\sin(3\cdot 0)\sin(\frac{\pi}{2}),\ \cos(3\cdot 0)\cos(\frac{\pi}{2}) \rangle = \langle 1, 0 \rangle$

(c) [1] Interpret your answer in (b) graphically and consider referencing the graph of f shown to the right.

(+.5)
The direction of steepest ascent is in the direction parallel to the x-axis

connect to graph (+.5)

(d) [3] Find the linear approximation of f at the point $(0, \frac{\pi}{2})$. stated (+.5)

We need to find the tangent plane - I'll use $z - z_1 = m_x(x - x_1) + m_y(y - y_1)$ where

eq of line (+.5)

connect to f_x, f_y interpret comp (+1)

$m_x = f_x(0, \frac{\pi}{2}) = 1$ $\Rightarrow$ $z - 1 = 1(x - 0) + 0(y - \frac{\pi}{2})$ } plug in (+.5)

$m_y = f_y(0, \frac{\pi}{2}) = 0$

or

(+.5) $f(0, \frac{\pi}{2}) = 0 + \cos(3\cdot 0)\sin(\frac{\pi}{2})$ $= 1\cdot 1 = 1$

$z = 1 + x - 0$
$= 1 + x$

6) [3] a) (+1) $f_x(1/3, 1) < 0$ neg c) Explain how you know: explain connection between $\hat{x}$ + graph (1)

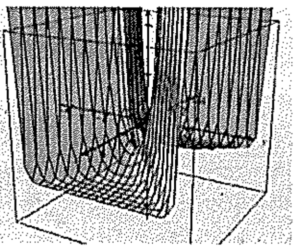
b) (+1) $f_y(1/3, 1) > 0$ pos

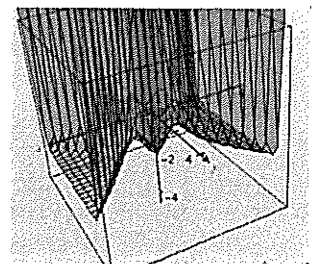
6.7 Consider the function $g(x,y) = 3(x^2+y^2)e^{y^2-x^2} - 2$. Three views of the function g are shown below. Identify all critical points and then classify them as local minimums, local maximums, or saddle points.

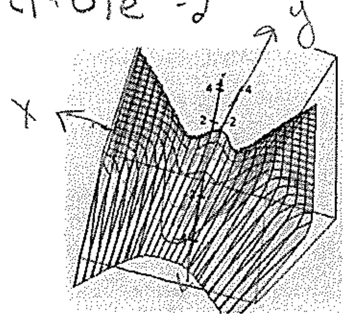
$g(-1,0) = 3(1+0)e^{0-1} - 2 = 3e^{-1} - 2$ $g(1,0) = 3(1+0)e^{0-1} - 2$

(+1) plan/outline

notation (.5)







upside down

(.5) $g_x(x,y) = 3(x^2+y^2)\,e^{(y^2-x^2)}(-2x) + 3\cdot 2x\, e^{y^2-x^2}$

(.5) $g_y(x,y) = 3(x^2+y^2)e^{(y^2-x^2)}(2y) + 3\cdot 2y e^{y^2-x^2}$

we'll set the 1st derivative to zero + solve for the critical points.

(.5) $g_x(x,y) = 0 = g_y(x,y)$

$\Rightarrow 3(x^2+y^2)e^{(y^2-x^2)}\,{-2x} + 6x e^{y^2-x^2} = 0$

$e^{y^2-x^2}\left[-6x(x^2+y^2) + 6x\right] = 0$

$e^{y^2-x^2} = 0$ or $6x\left[-(x^2+y^2)+1\right] = 0$

never happens

$1 - x^2 - y^2 = 0$ or $6x = 0$

$1 = x^2+y^2$ (:) $x = 0$

we're on the unit circle

factoring (+1)

(+1) knowledge (.5) alg

$3(x^2+y^2)e^{(y^2-x^2)}2y + 3\cdot 2y e^{y^2-x^2} = 0$

$e^{y^2-x^2}\left[6y(x^2+y^2) + 6y\right] = 0$

$e^{y^2-x^2} = 0$ or $6y\left[(x^2+y^2)+1\right] = 0$

$x^2+y^2+1 = 0$ never happens $y = 0$

if $y = 0$ then (:)

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

alg notation (.25) points

$\therefore$ our (x,y) value's for critical points: $(0,0)$, $(-1,0)$, $(1,0)$

Now we can either use the 2nd derivative test or inspect the graphs provided.

(+1) The points $(-1, 0, 3/e - 2)$ + $(1, 0, 3/e - 2)$ are saddle points

(.5) The (x,y) inputs $(0,0)$ gives a local min. on the graph of -2