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Exam 1

# Tmath 126

Summer ~~Spring~~ 2011

1. [8] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

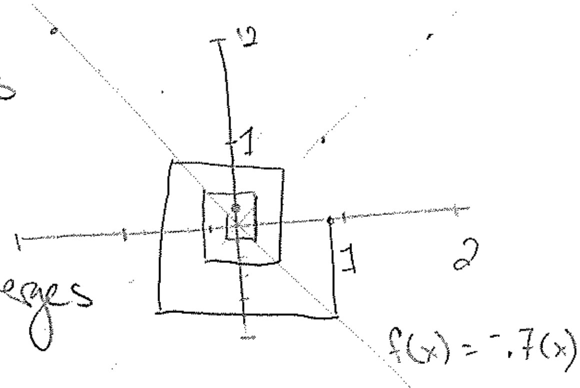
   (a) A sequence can be thought as an "infinite sum".

start (1.5)
truth value (1.5)
justify/counter ex (1)

False: a sequence is a list of things
ex $\{1,1,1,1,1,1,\ldots\}$
notice that the above example is not an 'infinite sum'

   (b) Let $r = -.7$, then the sequence $a_n = r^n$ converges.

true: Cobwebbing will verify this
$a_0 = 1$
$f(x) = -.7x$
limit is 0 $\therefore$ converges

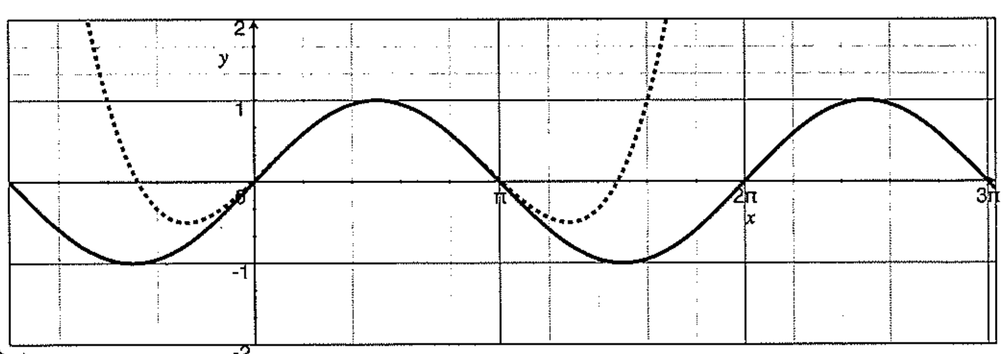



   (c) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

False: harmonic series $\sum 1/n$
note $\lim_{n\to\infty} 1/n = 0$ but $\sum 1/n$ diverges

   (d) The dotted function below is the 4$^{\text{th}}$ Taylor polynomial of $\sin(x)$ centered at 0.

False
although it may be the taylor poly centered at $\pi/2$



Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [4] Write the following sum using the sigma notation:

sigma +.5

$$-3+\frac{4}{5}+\frac{-1}{5}+\frac{6}{125}+\frac{-7}{625} = \frac{-3}{1}+\frac{4}{5}+\frac{-5}{25}+\frac{6}{125}+\frac{-7}{625}$$

+.5 +.5 +.5

$$\sum_{n=0}^{4}\frac{(-1)^{n+1}(n+3)}{5^n}$$

+.5 +.5 powers match +.5

signs match +.5

3. [12] Compute the following if possible.

$$\lim_{n\to\infty}\tan\left(\frac{4n\pi}{3+12n}\right)$$

stated +.5

$= \tan\left(\lim_{n\to\infty}\frac{4n\pi}{3+12n}\right)$ +.5

L'H +.5 $= \tan\left(\lim_{n\to\infty}\frac{4\pi}{12}\right)$ took right +.5

$= \tan\left(\lim_{n\to\infty}\frac{\pi}{3}\right)$

+.5

$= \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

end limit +.5

$$\sum_{n=1}^{\infty}\frac{3n^2-3}{n^2+4}$$

+.5 Does not exist

stated +.5

reasoning +1.0

b/c $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^2-3}{n^2+4}$

limit +.5

notation +.5

L'H $= \lim_{n\to\infty}\frac{6n}{2n} = 3$

b/c the terms do not converge to zero, the infinite sum can not converge to a finite #

$$\sum_{n=1}^{\infty}9(0.4)^{n-1}$$

$= 9 + 9\cdot.4 + 9\cdot.4^2 + \ldots$

+.5 Geometric series $\left.\begin{matrix}a=9\\ r=.4\end{matrix}\right\}$ +1

converges to

+.5 $\left\{\frac{a}{1-r} = \frac{9}{1-.4} = \frac{9}{.6}\right\}$ +.5

$= \frac{9}{\frac{6}{10}} = \frac{90}{6} = \frac{30}{2}$

$= 15$ +.5

$$\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n)!}$$

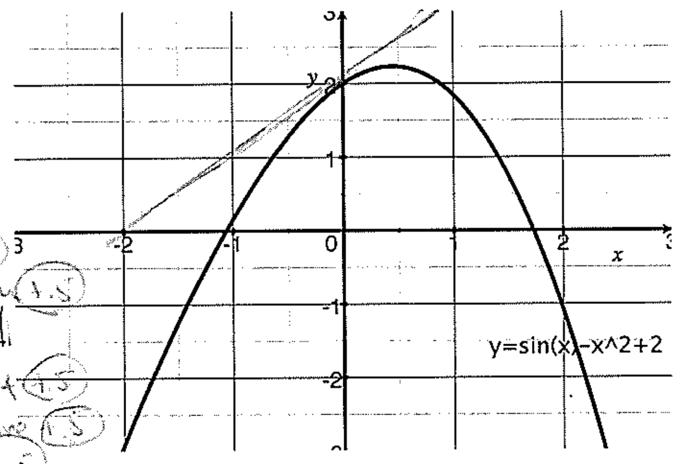
looks like $\underbrace{\cos}_{+1}\underbrace{\left(\frac{\pi}{6}\right)}_{+.5}$

+.5 $\left\{\frac{\sqrt{3}}{2}\right.$

stated +.5

looking at series we know +.5

4. Below is the graph of the function $f(x) = \sin(x) - x^2 + 2$.




(a) [3] Find the equation the line tangent to the graph of $f$ when $x = 0$.

find $y = mx + b$

$m = f'(0) = \cos(0) - 2(0) = 1 - 0 = 1$

thru $(0, 2)$ so

$y = +x + 2$

deriv (+.5) $f'(0)$ (+.5) $f'(0) = m$ (+.5) used point (+.5) eq of line (+.5) got it (+.5)

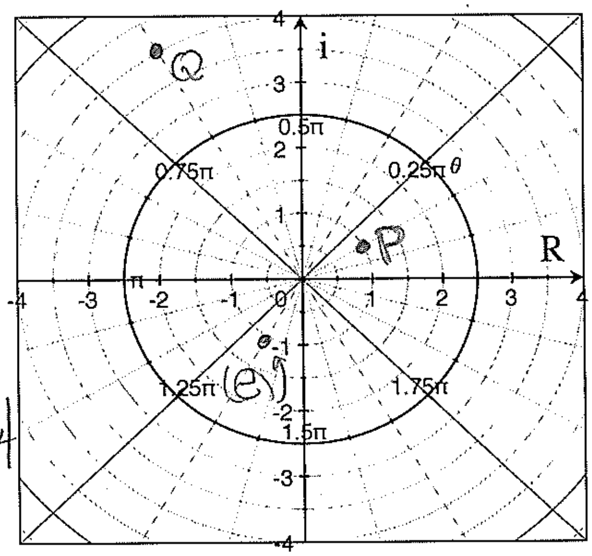
(b) [2] If Newton's method is used to locate a root of the equation $f(x) = 0$ and the initial approximation is $x_1 = 0$, find the second approximation $x_2$.

(+1) $\{$ the root of the line tangent to $f$ when $x = 0$

ie $0 = -x + 2 \Rightarrow x = 2$ $\}$ (+1)

5. Let $P$ and $Q$ be complex numbers where $P = e^{\frac{i\pi}{6}}$ and $Q = -2 + (2\sqrt{3})i$

(a) [1] Convert $P$ to rectangular coordinates (that is, of the form $a + bi$).




$e^{i\pi/6} = 1\cos\pi/6 + i\sin\pi/6 = \sqrt{3}/2 + i\frac{1}{2}$

$\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(+.5) (+.5)

(b) [1] Convert $Q$ to polar form (that is, of the form $re^{i\theta}$).

$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$

$= \sqrt{4 + 4 \cdot 3} = \sqrt{16} = 4$ (+.5)

$\sqrt{3}$ ? 2 θ sohcahtoa $\}$ (+.5)

$\tan ? = \frac{2\sqrt{3}}{2} = \sqrt{3}$ used correct (+.5)

$\Rightarrow ? = \pi/3 \Rightarrow \theta = 2\pi/3$ $4e^{i\frac{2\pi}{3}}$ (+.5)

(c) [1] Plot $P$ and $Q$ on the complex plane provided to the right.

(d) [1] Multiply $P \cdot Q$.

$(\sqrt{3}/2 + \frac{1}{2}i)(-2 + 2\sqrt{3}i)$ or $e^{i\pi/6} \cdot 4e^{i2\pi/3} = 4e^{i(\pi/6 + 2\pi/3)}$

$-\sqrt{3} + 3i - i + \sqrt{3}i^2$ $= 4e^{i5\pi/6}$

$-2\sqrt{3} + 2i$

(e) [2] Multiply $P^{20}$ and then plot this point on the axes provided on the right. (+.5)

$(e^{i\pi/6})^{20} = e^{i\frac{\pi}{6} \cdot 20} = e^{i\frac{10\pi}{3}} = e^{i\frac{4\pi}{3}}$

(+1) simplify (+.5)

6. Consider the function $\dfrac{2}{3-x}$.

(a) [3] Find a power series representation for the above function.

Recall $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$ (+.5)

$$\frac{2}{3-x} = \frac{2}{3} \cdot \frac{1}{1-\left(\frac{x}{3}\right)} = \frac{2}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \left(\frac{x}{3}\right)^3 + \left(\frac{x}{3}\right)^4 + \ldots\right]$$ use known results (+1)

(+1) or $\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$

note: this power series only represents the function $\frac{2}{3-x}$ as long as $-1 < \frac{x}{3} < 1$ or $-3 < x < 3$.

(+.5) trying to use known power series

(b) [3] Find the second order Taylor polynomial $T_2(x)$ based at $b = 0$.

(+1) Notice that the above is the Taylor Series

(+2) $\Rightarrow T_2(x) = \frac{2}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2\right]$

$= \frac{2}{3} + \frac{2x}{9} + \frac{2x^2}{27}$

or

(+.5) $f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2$

we can do the computation from scratch. plug in (+1)

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ |
|---|---|---|
| 0 | $2(3-x)^{-1}$ | $2/3$ |
| 1 | $2(3-x)^{-2}$ (+.5) | $2/9$ |
| 2 | $4(3-x)^{-3}$ (+.5) | $4/27$ |

(+.5)

(note this gives you the same answer.)

(c) [4] Bound the error $|f(x) - T_2(x)|$ on the interval $[-1.5, 1.5]$.

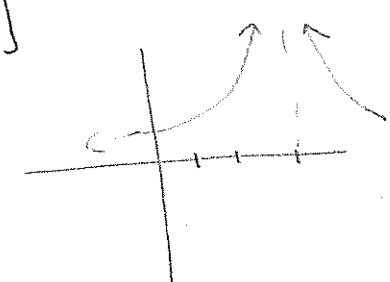
Recall error is bounded by $\frac{M}{3!}(x-0)^3$ (+1) formula. $n=3$ $b=0$ (+1)

where $|f^{(3)}(x)| < M \quad \forall x \in [-1.5, 1.5]$

find M (+.5)

justify M choice (+.5)

Notice $f^{(3)}(x) = \underbrace{12(3-x)^{-4}}_{(+.5)} = \frac{12}{(3-x)^4}$



Thus $f^{(3)}(x)$ reaches a max on the interval $[-1.5, 1.5]$ at 1.5.

Set $M = \frac{12}{(3-1.5)^4} = \frac{12}{1.5^4} = 2.37 \Rightarrow$ error $< \frac{2.37}{3!}(1.5-0)^3$

$= 1.33$ got .1 (1.5)