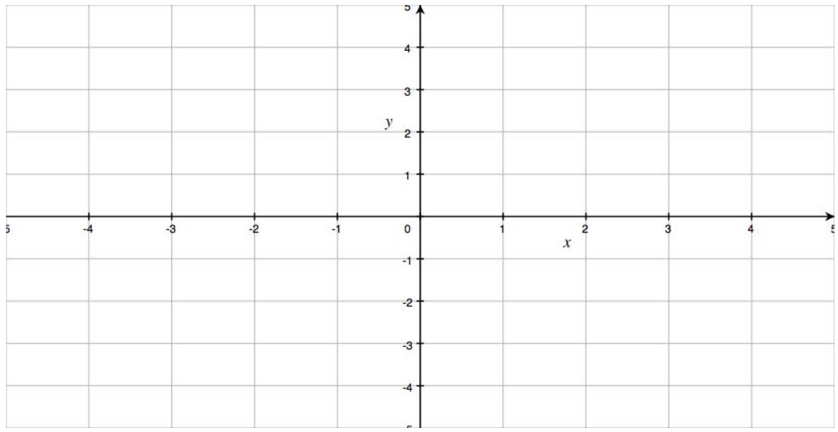
Cobwebbing Sequences

Adapted from homework created by Jonny Comes.

1. Let $f(x) = \frac{x^2}{2}$. Carefully sketch the graph of f on the right.



Suppose $a_n = f(a_{n-1})$ for all integers $n > 0$. Use cobwebbing techniques to investigate the sequence $\{a_n\}_{n=1}^{\infty}$.

(a) Write down the sequence if $a_1 = 1$.

(b) If $a_1 = 1$, does the the sequence converge? If so, what to?

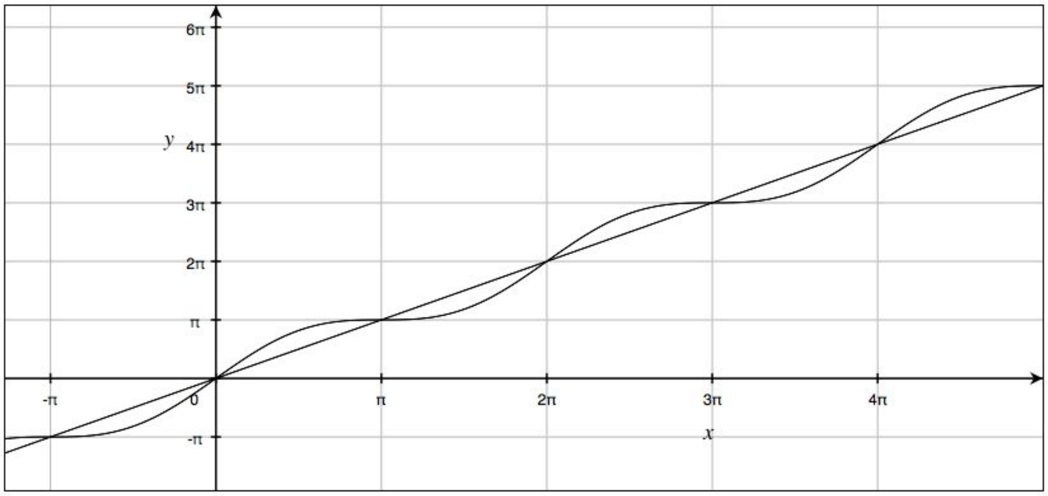
(c) Write down the sequence if $a_1 = -1$.

(d) If $a_1 = -1$, does the the sequence converge? If so, what to?

(e) For which values of a_1 is the sequence $\{a_n\}$ converging?

(f) For each value of a_1 that makes the sequence $\{a_n\}$ converge, state the limit.

2. Pictured below are the graphs of $y = \sin(x) + x$ and $y = x$.



Suppose $a_n = \sin(a_{n-1}) + a_{n-1}$ for all integers $n > 0$.

(a) For which values of a_1 does the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0?

(b) For which values of a_1 does the sequence $\{a_n\}_{n=1}^{\infty}$ converges to π?

(c) For which values of a_1 does the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 2π?

(d) For which values of a_1 does the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 3π?

3. Suppose r is a real number and $a_n = ra_{n-1}$ for all integers $n > 0$.

   (a) Consider the special case when $r = 0$. Let $a_1 = 0$ and write down a few terms of the sequence $\{a_n\}_{n=1}^{\infty}$ and determine if the sequence converges.

   (b) Consider a second special case when $r = 1$. Let $a_1 = 1$ and write down a few terms of the sequence $\{a_n\}_{n=1}^{\infty}$ and determine if the sequence converges.

   (c) Assume $r = 2$. What functions should you plot to make use of cobwebbing? Use cobwebbing to find all values of a_1 for which the sequence $\{a_n\}$ converges.

   (d) Assume $r > 1$. What functions should you plot to make use of cobwebbing? Use cobwebbing to find all values of a_1 for which the sequence $\{a_n\}$ converges.

   (e) Repeat part (d) for $0 < r < 1$.

   (f) Collect your results in parts (d) and (e) to finish the following sentence:
   Let $0 \leq r$, then the sequence $a_n = r^n$ converges if

   (g) Repeat part (d) for $r = -1$.