1
10
15
24
14
65

FINAL **TMath 126** /65 Spring 2017

You may use:

- any kind of calculator that cannot access the internet and
- 3 double-sided 3 × 5" cards for this exam.

1. [1] (prerequisite material) Your Name: Key

2. [9] Identify each statement below as true if the statement is *always* True and provide a brief justification. Otherwise, identify the statement as False and provide a counterexample or brief justification.

   (a) (practice final #1) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that the n^{th} partial sum of the associated series is $s_n = \dfrac{n+n^2}{n^2-\pi}$. Then $\lim_{n\to\infty} a_n = 0$.

   start (+.5)
   answer (+1)
   true (+.5)
   justify (+1)

   Note $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n+n^2}{n^2-\pi} \overset{L'H}{=} \lim_{n\to\infty} \frac{1+2n}{2n} \overset{L'H}{=} 1$

   The series converges to a finite # thus the terms a_n must be converging to zero

   TRUE

   (b) (exam2 #1) If $\vec{w}(t)$ and $\vec{v}(t)$ are vector-valued functions, then $(\vec{v}'(3) \cdot \vec{w}''(3)) + \vec{v}(2)$ returns a vector valued function.

   $\vec{v}'(3) \cdot \vec{w}''(3)$ returns a number/scalar
   We can't add number to vectors (like $\vec{v}(2)$)

   FALSE

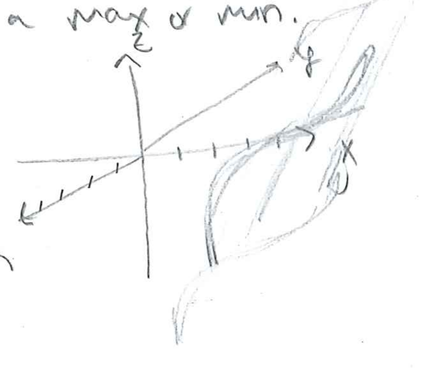
   (c) (quiz4 #1) If f is a function such that $f_x(4,-2) = 0$ and $f_y(4,-2) = 0$ then $f(4,-2)$ is a maximum or a miniumum.

   FALSE

   We know $(4,-2,f(4,-2))$ is a Critical Point, it may or may not be a max or min.

   ex $z = (x-4)^3$

   flat tangent plane but not max or min



Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

3. The number of people in line is taken every hour but only the last five readings are recorded. Below is a chart of the data N (number of people) at t o'clock. From this we can estimated the first and second derivatives of N at $t = 11$.

t	9	10	11	12	13
$N(t)$	10	7	8	22	12

n	0	1	2
$N^n(11) \approx$	8	14	13

(a) [5] (HW2 §9.7 #1) Use the above approximate derivatives and data to estimate for the number of people in line at time t where t is close to 11.

StrA (+.5)

(+1) We can use 2nd degree Taylor Approximation

(+1) Since we "know" $N^n(11)$, we'll center @ $11=t$

(+.5) (+.5)

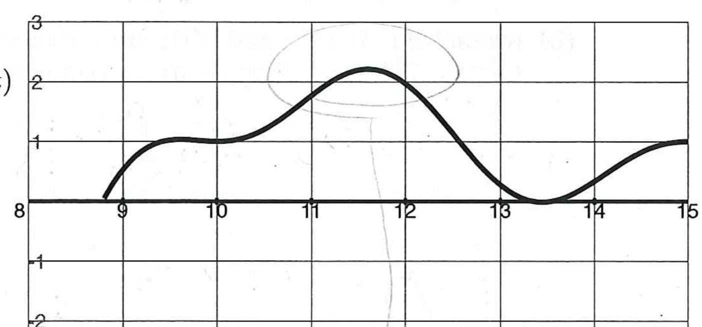
$$N(t) \approx N(11) + \frac{1}{1!}(t-11)N'(11) + \frac{1}{2!}(t-11)^2 N^{(2)}(11)$$

(+.5)

$$N(t) \approx 8 + 14(t-11) + \frac{13}{2}(t-11)^2$$

(+.5) (+.5)

(b) [3] (WebHW5 #1) The length of the line changes rather slowly & experimentally we know $N^{(3)}(t)$ has the following graph. Provide an upper bound for the error if we use the above estimate of $N(14)$.



for (+.5)

(+.5) (+1)

$$\text{Error}(t) \le \frac{1}{3!}(t-11)^3 \max|N^{(3)}(z)|$$

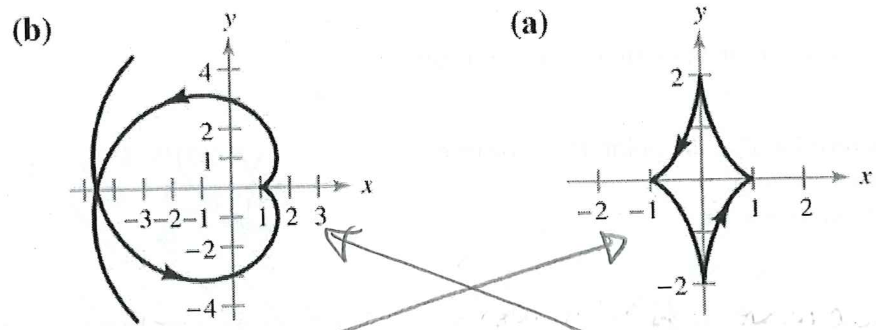
where z is between 8 and 14

$$\text{Error}(14) \le \frac{1}{3!}\underbrace{(14-11)^3}_{(+.5)} \cdot 2.5 \quad (+.5)$$

$$\frac{\cancel{3}\cdot 3\cdot 3}{\cancel{3}\cdot 2} \cdot 2.5 = 4.5 \cdot 2.5 = 11.25$$

4. [4] (HW5 §10.2 #1) Match the set of parametric equations with their graph. *Justify* yourself.

ID (+2)



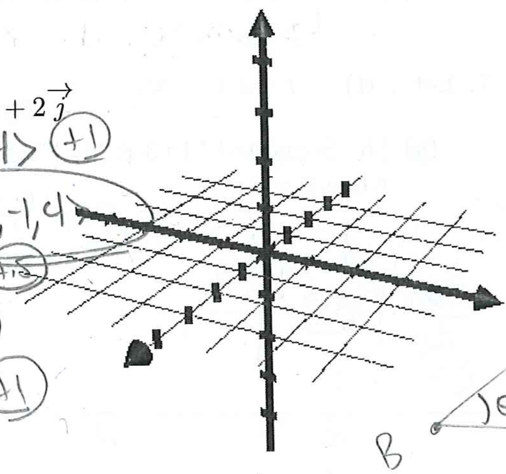
i) $x(t) = \cos^3(t)$ and $y(t) = 2\sin^3(t)$

note the x value stays between -1 & 1
the y-value stays between -2 & 2 (+1)

ii) $x(t) = \cos(t) + t\sin(t)$ and $y(t) = \sin(t) - t\cos(t)$.

process of elimination
(+1)

5. Consider the points $A(0, 0, 4)$, $B(2, 3, 0)$ and $C(1, -2, 1)$.

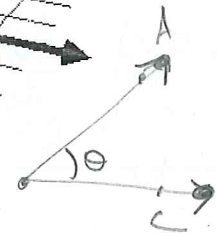


(a) [3] (Exam2 #2) Find the components of $\overrightarrow{BA} + 2\vec{j}$

$\overrightarrow{BA} = \langle 0-2, 0-3, 4-0\rangle = \langle -2, -3, 4\rangle$ (+1)

$\overrightarrow{BA} + 2\vec{j} = \langle -2,-3,4\rangle + \langle 0,2,0\rangle = \langle -2,-1,4\rangle$ (+1.5) (+.5)

(b) [4] (Exam2 #2) Find the angle $\angle ABC$.



$\overrightarrow{BC} = \langle 1-2, -2-3, 1-0\rangle = \langle -1,-5,1\rangle$ (+1)

Recall $\cos\theta = \dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\|\overrightarrow{BA}\|\cdot\|\overrightarrow{BC}\|}$ (+1)

$\Rightarrow \theta = \arccos\left(\dfrac{2+15+4}{\sqrt{4+9+16}\,\sqrt{1+25+1}}\right) = \arccos\left(\dfrac{21}{\sqrt{29}\sqrt{27}}\right) \approx 41^\circ$ or $.71558$ rad (+.5) (+.5) (+1)

Start (+.5) (c) [4] (WebHW8 #6) Find the equation of the plane that passes through A, B, and C.

Note the normal $\vec{n} = \overrightarrow{BA} \times \overrightarrow{BC}$ (+.5)

$$\begin{vmatrix} i & j & k \\ -2 & -3 & 4 \\ -1 & -5 & 1 \end{vmatrix} = \vec{i}(-3+20) - \vec{j}(-2+4) + \vec{k}(10-3)$$ or

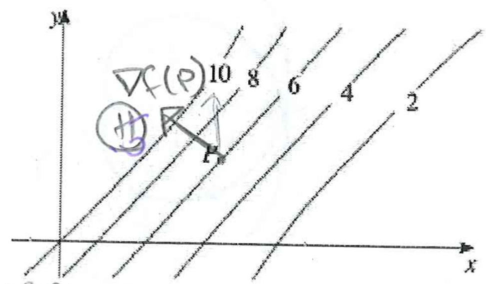
$= 17\vec{i} - 2\vec{j} + 7\vec{k}$ complete (+1)

So

(+1) $\vec{n}\cdot(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle) = 0$

$\Rightarrow \langle 17,-2,7\rangle \cdot (\langle x,y,z\rangle - \langle 0,0,4\rangle) = 0$ (+.5)

6. Let f have the contour lines shown on the right.



(a) [3] Determine if f_y at point P is positive, negative, or zero.
Justify your answer.

(+1) [$f_y(P)$ is the rate of change of z as we move // to the y-axis,
So as we move up, the level curves increase in quantity $\Rightarrow$ f_y is positive (+1)

Sketch (+.5)
true (+.5)

(b) [2] Sketch direction of the vector $\nabla f(P)$ on the graph.

↳ direction of steepest ascent (+1)

a vector (+.5)

7. Let $\vec{r}(t) = \langle t+4, t^3 - 3t \rangle$.
↳ in $\mathbb{R}^2$

(a) [4] (Suggested §10.3 #25) Find the equation of the line tangent to the graph of $\vec{R}(t)$ when $x = 2$

looking for $y - y_1 = m(x - x_1)$ (+1)

$m = \frac{dy}{dx}\Big|_{x=2}$ (+.5) $= \frac{dy/dt}{dx/dt}\Big|_{x=2}$ (+.5) $=$ (+.5) $\frac{3t^2-3}{1}\Big|_{x=2}$

when $x=2 \Rightarrow 2 = t+4 \Rightarrow t = -2$ (+.5)

$m = \frac{3(-2)^2 - 3}{1} = 9$ (+.5)

when $t=-2$, $x=2$, $y = -8+6 = -2$ (+.5)

$y + 2 = 9(x-2)$

or

$x = t+4 \Rightarrow t = x-4$
Sub in for $y = t^3 - 3t$
$\Rightarrow y = (x-4)^3 - 3(x-4)$ (+1)

looking for $y - y_1 = m(x - x_1)$ (+1)

$m = \frac{dy}{dx}\Big|_{x=2}$ (+.5) $= 3(x-4)^2 - 3\Big|_{x=2}$ (+.5) $= 9$ (+.5)

(+.5) when $x = 2$ then $y = (2-4)^3 - 3(2-4) = -2$

$y + 2 = 9(x-2)$

(b) [3] (Exam2 #3) Find $\int \vec{r}(t)\,dt$.

(+.5) integrate each piece

one integral correct (+.5)

$\langle \underbrace{\tfrac{1}{2}t^2 + 4t + C_1}_{(+.5)}, \underbrace{\tfrac{1}{4}t^4 - \tfrac{3}{2}t^2 + C_2}_{(+.5)} \rangle$

or

$\langle \tfrac{1}{2}t^2 + 4t, \tfrac{1}{4}t^4 - \tfrac{3}{2}t^2 \rangle + \vec{C}$

constant (+1)

8. [9] (5/18 Lecture) Choose *ONE* of the following. Clearly identify which of the two (a or b) you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.
For either situation a xor situation b, you need to:

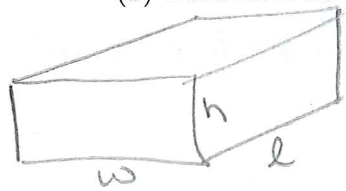
- identify the function you need to optimize and
- outline the steps needed to find the minimums and verify this is the minimum. Make sure to highlight any steps involving calculus or precalculus techniques. You do not need to perform the steps but you do need to make sure that your process works.

BONUS: (up to 3 points) if you outline a second method to find the minimum!

(a) A home improvement contractor is painting the walls and ceiling of a rectangular room. The volume of the room is 998.25 ft^3. The cost of paint is \$0.12 per ft^2 and the ceiling paint cost is \$0.18 per ft^2. Find the room dimensions that will minimize costs.

(b) Find three numbers whose sum is 42 but whose sum of squares is minimized.

(a)



start (+.5) variables (+1)

Volume $= w \cdot h \cdot l = 998.25\ ft^3$ (+.5)

Cost $= .12 \cdot$ wall area $+ .18 \cdot$ ceiling area

Cost $= .12(2wh + 2lh) + .18(wl)$ (+1)

↳ function we want to minimize

(+2) 1) Need to reduce Cost to function of just 2 variables
→ solve for h in Volume eq
→ sub. this into the Cost function.

(+2) 2) Find the Critical Points (CP)
→ Find w and l so that $\frac{\partial Cost}{\partial l} = 0$ and $\frac{\partial Cost}{\partial w} = 0$ or not defined
→ Note we may need Newton's method to find the roots?

(+1) 3) Use the 2nd Derivative Test on each CP to determine min

(+1) notation/clear

start (+.5) variables (+1)

(b) Label the 3 #'s x, y and z

Want $x+y+z=42$ (+.5)

S = Sum of squares $= x^2+y^2+z^2$ (+1)

↳ function we want to minimize

1) Use Lagrange multipliers (+1)
→ set up equation of 4 unknowns & 4 eq

let $S = x^2+y^2+z^2$, $C = x+y+z$ (+1)

(+1) $\left\{ \begin{array}{l} \nabla S = \lambda \nabla C \\ C = 42 \end{array} \right\} = \left\{ \begin{array}{l} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x+y+z=42 \end{array} \right\}$

(+1) → Use a series of substitutions to solve for x, y and z (we don't really need λ)

$2x=2y \Rightarrow x=y \quad 2y=2z \Rightarrow y=z$

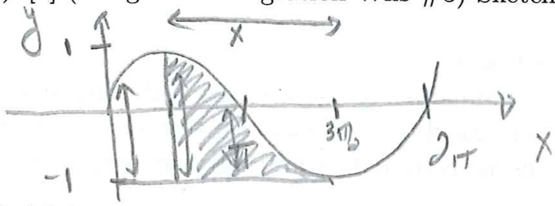
$\Rightarrow 3x=42 \Rightarrow x=14=y=z$

(+1) 2) Verify we have a minimum by testing points

(+1) notation/clear

9. Consider $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{-1}^{\sin(x)} f(x,y)\,dy\,dx$

(a) [3] (Integrated Integration Wks #3) Sketch the region of integration in the xy plane.



graph sin (+.5)
y bounds (+1)
x bounds (+1)
got it (+.5)

(b) [3] (WebHW17 #10) Switch the order of integration.

x varies between $y=\sin(x)$ and $x=\frac{\pi}{2}$

$\Rightarrow$ between $x=\arcsin(x)$ and $x=\frac{\pi}{2}$

solve for x (+.5)

will need to modify from standard one

so $\arcsin(x)$ defined as $\arcsin(x)+\frac{\pi}{2}$

normally $[-\frac{\pi}{2}, \frac{\pi}{2}]$

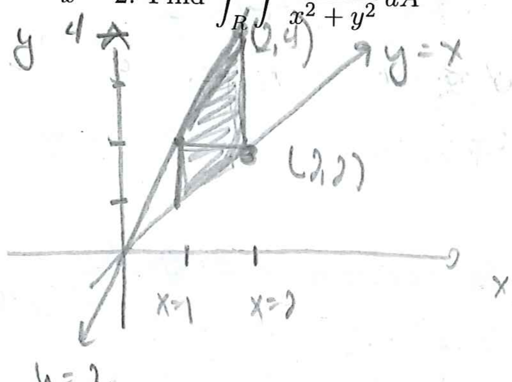
now want $[\frac{\pi}{2}, \frac{3\pi}{2}]$

y varies between -1 & 1

$$\int_{-1}^{1}\int_{\frac{\pi}{2}}^{\arcsin(x)+\frac{\pi}{2}} f(x,y)\,dx\,dy$$

(+1) (+1) switch (+.5)

10. [5] (Suggested §14.2 #13) Let R be the trapezoid bounded by $y=x$, $y=2x$, $x=1$, and $x=2$. Find $\iint_R \frac{y}{x^2+y^2}\,dA$



draw (+1)

(+1) (+1) (+.5)

$$\int_1^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx$$

$u = x^2+y^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$\int \frac{y}{x^2+y^2}\,dy = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(u)+c$$

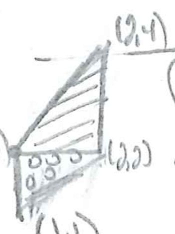
(+1.5)

$$\int_1^2\int_x^{2x} \frac{y}{x^2+y^2}\,dy\,dx = \int_1^2 \frac{1}{2}\ln|x^2+y^2|\Big]_x^{2x}\,dx = \int_1^2 \frac{1}{2}\ln(x^2+(2x)^2) - \frac{1}{2}\ln(x^2+x^2)\,dx$$

$$= \int_1^2 \frac{1}{2}\ln(5x^2) - \frac{1}{2}\ln(2x^2)\,dx = \frac{1}{2}\int_1^2 \ln\frac{5\cancel{x^2}}{2\cancel{x^2}}\,dx = \frac{1}{2}\int_1^2 \ln\left(\frac{5}{2}\right)dx$$

$$= \frac{1}{2}\ln\left(\frac{5}{2}\right)x\Big]_1^2 = \frac{1}{2}\ln\left(\frac{5}{2}\right)$$

or



$$\int_2^4\int_{\frac{1}{2}y}^{2} \frac{y}{x^2+y^2}\,dx\,dy + \int_1^2\int_1^{y} \frac{y}{x^2+y^2}\,dx\,dy$$

draw (+1)
broke up (+1)
bounds (+1)
form (+.5)
integration (+1.5)