Key

Exam 1 **Tmath 126** Spring 2011

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, circle F and provide a counterexample.

   (a) If there exists some number M such that $a_n \leq M$ for all n, then $\{a_n\}$ converges.

   False let $a_n = -n$ then $a_n \leq 1$ for all n but

   $\{-1, -2, -3, -4, -5, ..\}$ does not converge

   (b) The Taylor series is an example of a power series.

   True power series are 'polynomials with an infinite # of terms' which Taylor series are

   (c) Given a function f, the associated taylor series T has the property that $f(x) = T(x)$ for all x.

   False $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ..$ but only if $x \in (-1, 1)$

   (d) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

   True since $\cos(x) + i\sin(x) = e^{ix} = 1 + ix - \frac{x^2}{2!} + i\frac{x^3}{3!} - \ldots$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [4] Write the following sum in expanded form:

$$\sum_{n=1}^{4} \frac{\sqrt{2n+1}}{n!}$$

$$\frac{\sqrt{2+1}}{1!} + \frac{\sqrt{4+1}}{2!} + \frac{\sqrt{6+1}}{3!} + \frac{\sqrt{8+1}}{4!} = \frac{\sqrt{3}}{1} + \frac{\sqrt{5}}{2} + \frac{\sqrt{7}}{6} + \frac{3}{8}$$

3. [4] Write the following sum using the sigma notation:

$$1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \frac{5}{81}$$

$$\sum_{n=0}^{4} (-1)^n \frac{n+1}{3^n}$$

(note: there are many right answers for this question)

4. [20] Compute the following if possible.

$\lim_{n\to\infty} \frac{9^{n+1}}{10^n} = \lim_{n\to\infty} 9\left(\frac{9}{10}\right)^n$

$\{9(9/10), 9\cdot(9/10)^2, 9(9/10)^3, \ldots\}$

looks like $a_n = 9r^n$

where $r = 9/10 < 1$

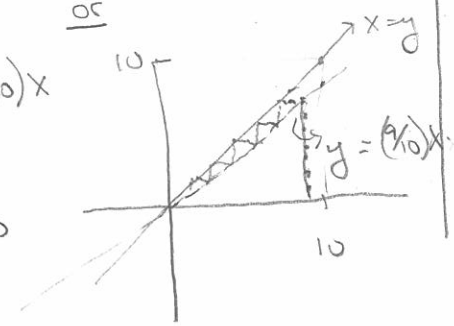
$\therefore$ seq will converge to zero

or

$f(x) = (9/10)x$

$a_1 = 9$

conv. to zero



$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ this one is a telescope hiding?

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$\Rightarrow A(n+1) + B(n) = 1$

$\Rightarrow \begin{cases} An + Bn = 0 \\ A = 1 \end{cases} \Rightarrow \begin{cases} B = -1 \\ A = 1 \end{cases}$

$= \sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{-1}{n+1}\right)$ note that

$S_1 = \frac{1}{1} - \frac{1}{2} = 1 - 1/2$

$S_2 = \frac{1}{1} - \cancel{1/2} + \cancel{1/2} - 1/3 = 1 - 1/3$

$S_3 = 1/1 - \cancel{1/2} + \cancel{1/2} - \cancel{1/3} + \cancel{1/3} - 1/4 = 1 - 1/4$

$\Rightarrow S_n = 1 - 1/(n+1)$

$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1} + \frac{1}{2!} - \frac{1}{3!} + \ldots$

Recall $e^x = 1 + \frac{1}{1!}x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

if we let $x = -1$ the above series would match the series given above

thus

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}$

$\sum_{n=1}^{\infty} \frac{2}{n} = 2\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}$

harmonic series (known to diverge)

Diverges

or

$2[1 + 1/2 + \underline{1/3 + 1/4} + 1/5 + 1/6 + 1/7 + 1/8 + \ldots$

$\geq 2[1 + 1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + \ldots$

$= 2[1 + 1/2 + 1/2 + 1/2 + \ldots$

which diverges b/c $\lim_{n\to\infty} a_n \neq 0$.

5. Let $p(x) = x^6 - 5x^2 + 3x - 2$.

(a) [5] Find the second order Taylor polynomial $T_2(x)$ based at $b = 1$.

k	$p^{(k)}(x)$	$p^{(k)}(1)$
0	x^6-5x^2+3x-2	-3
1	$6x^5-10x+3$	-1
2	$30x^4-10$	20

$$p(1) + \frac{p'(1)}{1!}(x-1) + \frac{p''(1)}{2!}(x-1)^2$$

$$= -3 - (x-1) + \frac{20}{2}(x-1)^2$$

$$= -2 - x + 10(x-1)^2$$

(b) [5] Bound the error $|p(x) - T_2(x)|$ on the interval [0.5,1.5].

note $p^{(3)}(x) = 120x^3$

and the cubic is inc $[.5, 1.5]$

so we can bound $p^{(3)}(x)$

with $p^{(3)}(1.5) = 120\cdot(1.5)^3$

$$\text{error} \le \frac{m}{3!}|x-1|$$

$$\le \frac{120\cdot(1.5)^3}{3!}\cdot .5$$

$$= \frac{40(1.5)^3}{2}\cdot\frac{1}{2}$$

$$= 10(1.5)^3$$

6. [10] Find the Taylor series expansion for $\dfrac{x}{4+x}$ centered at 0, and find out where it converges.

I'd like to make use of the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$ which conv as long as $x \in (-1, 1)$.

So $$\frac{x}{4+x} = \frac{x}{4}\cdot\frac{1}{1+\frac{x}{4}} = \frac{x}{4}\cdot\frac{1}{1-(-x/4)}$$

$$= \frac{x}{4}\left[1 + \left(\frac{-x}{4}\right) + \left(\frac{-x}{4}\right)^2 + \left(\frac{-x}{4}\right)^3 + \left(\frac{-x}{4}\right)^4 + \dots\right]$$

$$= \frac{x}{4}\left[1 - \frac{x}{4} + \frac{x^2}{4^2} - \frac{x^3}{4^3} + \frac{x^4}{4^4} + \dots\right]$$

$$= \frac{x}{4} - \frac{x^2}{4^2} + \frac{x^3}{4^3} - \frac{x^4}{4^4} + \frac{x^5}{4^5} + \dots \quad \text{or} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$$

Conv. when

$$-1 < \frac{-x}{4} < 1$$

$$\Rightarrow -4 < -x < 4$$

$$\Rightarrow 4 > x > -4$$

7. [10] Compute the following indefinite integral.

$$\int \frac{\sin(x)}{x}\,dx = \int \frac{1}{x} \cdot \sin(x)\,dx$$

$$= \int \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right] dx$$

$$= \int 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots \; dx$$

$$= C + x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!} - \frac{x^{11}}{11\cdot 11!} + \cdots$$

8. [10] Use geometric series to show $0.\overline{99999}\ldots = 1$.

Note $.9\overline{99} = .9 + .09 + .009 + .0009 + \cdots$

$$= 9\left(\tfrac{1}{10}\right) + 9\left(\tfrac{1}{10}\right)^2 + 9\left(\tfrac{1}{10}\right)^3 + 9\left(\tfrac{1}{10}\right)^4 + \cdots$$

notice this is a geometric series where

$a = .9$ and $r = 1/10$ b/c $|r| < 1$,

the series converges to

$$\frac{a}{1-r} = \frac{.9}{1 - 1/10} = \frac{.9}{.9} = 1$$