3
~~4~~6
10 : 18

28

Exam 2 Key

Tmath 126

Spring 2011

1. [12] TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

Let $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ be vectors in $\mathbb{R}^3$.
Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

(a) $(\overrightarrow{a} \cdot \overrightarrow{b}) \cdot \overrightarrow{c} = \overrightarrow{a} \cdot (\overrightarrow{b} \cdot \overrightarrow{c})$.

stated +.5
answer +.5
+1 justification/counterex
+1 sence/notation

False, $\vec{a}\cdot\vec{b}$ is a number + a number cannot be doted with another vector $\vec{c}$.
Thus $(\vec{a}\cdot\vec{b})\cdot\vec{c}$ makes no sense.

(b) $||\overrightarrow{a} \times \overrightarrow{b}|| = ||\overrightarrow{b} \times \overrightarrow{a}||$.

True, The length of the vector $\vec{a}\times\vec{b}$ is equal to the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.
That area is the same as the parallelogram with side lengths $\vec{b}$ and $\vec{a}$.

(c) If $f(x, y)$ is a continuous function, the first-order derivatives exist, and $f_x(0, 0) = 0 = f_y(0, 0)$, then f has a local minimum or maximum at the point $(0, 0)$.

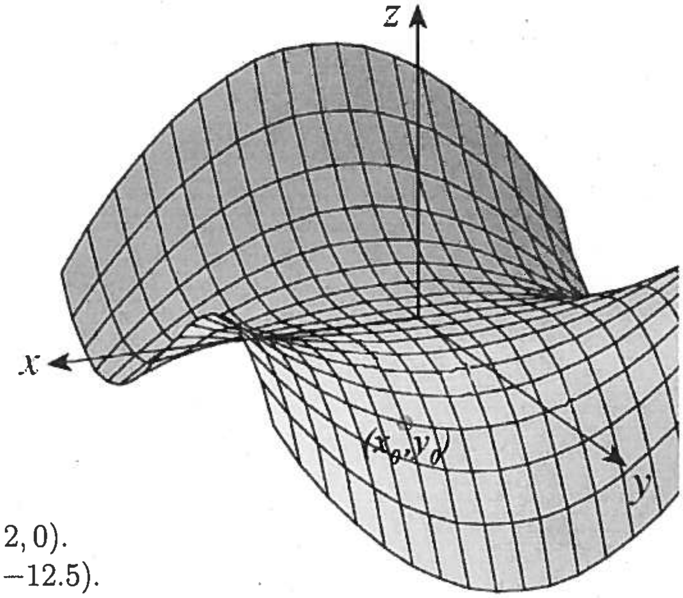
False, f may have a saddle at $(0,0)$.
For example $f(x,y)=x^2y$ shown in problem 8.

(d) If $f(x, y)$ is a continuous function, the first-order derivatives exist, and f has a local minimum or maximum at the point $(0, 0)$, then $f_x(0, 0) = 0 = f_y(0, 0)$.

True, this is the 3D generalization of the 2D version.
The peaks (max) + valleys (min) will have level tangent planes $\Rightarrow f_x(0,0)=0$
and $f_y(0,0)=0$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the function g pictured to the right and answer the following multiple choice questions.



   (a) [1] $g_x(x_0, y_0)$ is
      [✓ i.] > 0
      ii. < 0

   (b) [1] $\left.\dfrac{\partial g}{\partial y}\right|_{(x_0,y_0)}$ is
      i. > 0
      [✓ ii.] < 0

3. Let $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$. Also let $S(3,6,1.5)$ and $T(-9,-14,-12.5)$.

   (a) [3] Find the equation of the ~~line~~ plane that passes through P, R, and Q.

$\overrightarrow{PQ} = \langle 3-1, -1-3, 6-2\rangle = \langle 2,-4,4\rangle$
$\overrightarrow{PR} = \langle 5-1, 2-3, 0-2\rangle = \langle 4,-1,-2\rangle$ (+.5)

(+1)

$$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = i\begin{vmatrix} -4 & 4 \\ -1 & -2 \end{vmatrix} - j\begin{vmatrix} 2 & 4 \\ 4 & -2 \end{vmatrix} + k\begin{vmatrix} 2 & -4 \\ 4 & -1 \end{vmatrix}$$

$$= i(8 - {}^-4) - j(-4-16) + k(-2 - {}^-16)$$

$$= 12\vec{i} + 20\vec{j} + 14\vec{k}$$

thus $\vec{\eta} = \langle 12,20,14\rangle$

(+.5) eq of line

so the following would work:

$0 = \langle 12,20,14\rangle \cdot (\langle x,y,z\rangle - \langle 1,3,2\rangle)$

$0 = \langle 12,20,14\rangle \cdot (\langle x-1, y-3, z-2\rangle)$

(+.5) plug in into eqn

$0 = 12(x-1) + 20(y-3) + 14(z-2)$

$100 = 12x + 20y + 14z$

(+.5) stated

12
60
28

   (b) [2] Does the line that passes through S and T intersect the plane you found in part (a)? Justify your answer.

note $\overrightarrow{TS} = \langle 3 - {}^-9, 6 - {}^-14, 1.5 - {}^-12.5\rangle$
$= \langle 12,20,14\rangle$ (+.5)

thus $\overrightarrow{TS} \parallel \vec{\eta}$ and $\vec{\eta}$ is $\perp$ to the plane containing P, R & Q.
Thus $\overrightarrow{TS}$ must intersect the plane.

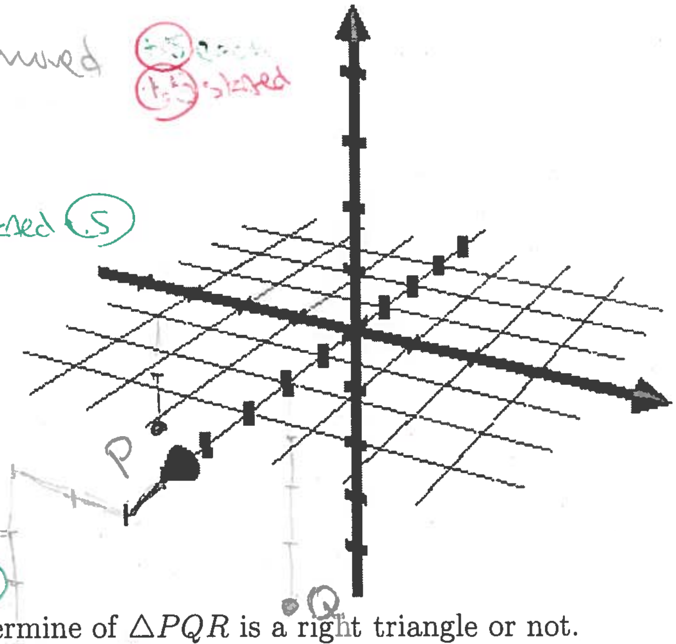
reasoning (+1) stated (+.5)

4. Consider the points: $P(1,-3,-2)$, $Q(2,0,-4)$, and $R(6,-2,-5)$.

(a) [2] Plot the points P, Q, and R.

Sorry, I should have moved the grid up higher.

(+.5) (+.5) sketched



(b) [3] Find the length of $\overrightarrow{PR}$ sketched (.5)

$\overrightarrow{PR} = \langle 6-1, -2-{-3}, -5-{-2} \rangle$

$= \langle 5, 1, -3 \rangle$ (+.5)

$\|\overrightarrow{PR}\| = \sqrt{5^2+1^2+(-3)^2}$ (+1.5)

$= \sqrt{25+1+9} = \sqrt{35}$

got it/alg (+.5)

(c) [4] Use calculus methods to determine of $\triangle PQR$ is a right triangle or not.

We need to find the angles between the 3 pairs of vectors.

(1) Recall $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ where θ is the angle between $\vec{u}$ & $\vec{v}$

R (+1) If $\vec{u}\cdot\vec{v} = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$

$\overrightarrow{PR} = \langle 5, 1, -3 \rangle$

$\overrightarrow{PQ} = \langle 1, 3, -2 \rangle$ (+.5)

$\overrightarrow{QR} = \langle 4, -2, -1 \rangle$

Start (+.5)

(+.5) So $\overrightarrow{PR}\cdot\overrightarrow{PQ} = 5\cdot1 + 1\cdot3 + {-3}\cdot{-2} \neq 0$

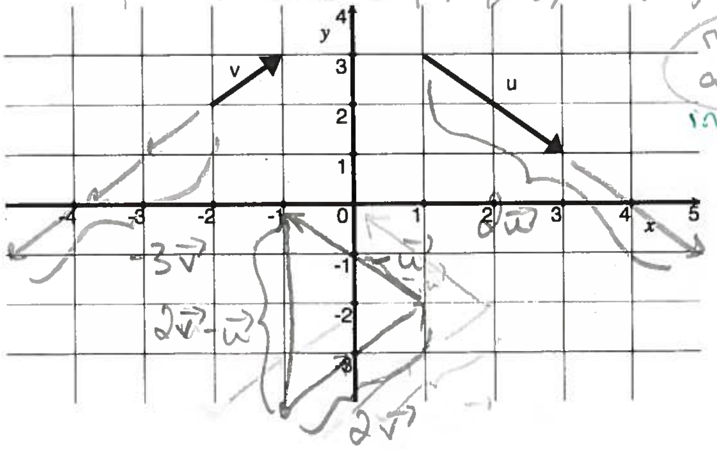
$\overrightarrow{PR}\cdot\overrightarrow{QR} = 5\cdot4 + 1\cdot{-2} + {-3}\cdot{-1} \neq 0$

$\triangle$ $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 4\cdot1 + 3(-2) + (-2)(-1) = 0$

right ∠ at Q

interpret (+.5)

5. Consider the vector $\vec{v}$ and $\vec{u}$ shown to the right.



(a) [1] Draw the vector $2\vec{u}$.

(b) [1] Draw the vector $-3\vec{v}$.

(c) [2] Draw the vector $2\vec{v} - \vec{u}$.

(.5) (.5)

(+1) added/sub

note

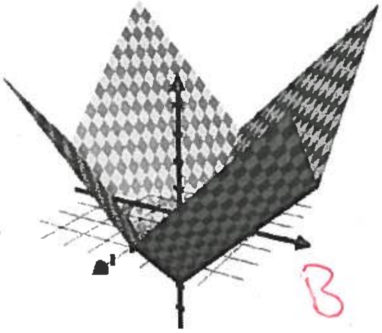
$2\vec{v} - \vec{u}$

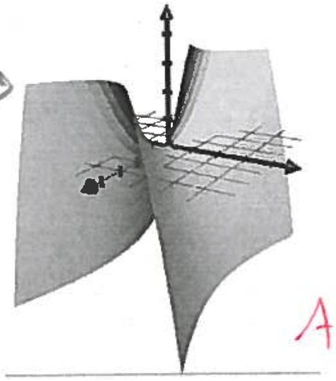
$= 2\langle -1, 1 \rangle - \langle 2, -2 \rangle$

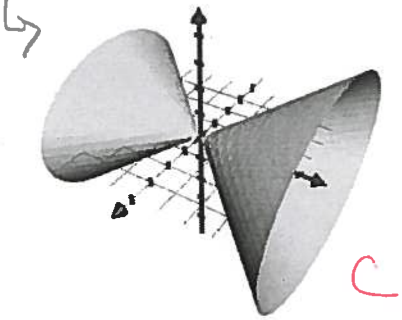
$= \langle -2-2, 2+2 \rangle = \langle 0, 4 \rangle$ so consistent with picture.

6. [3] Match the following equations to their respective graphs:

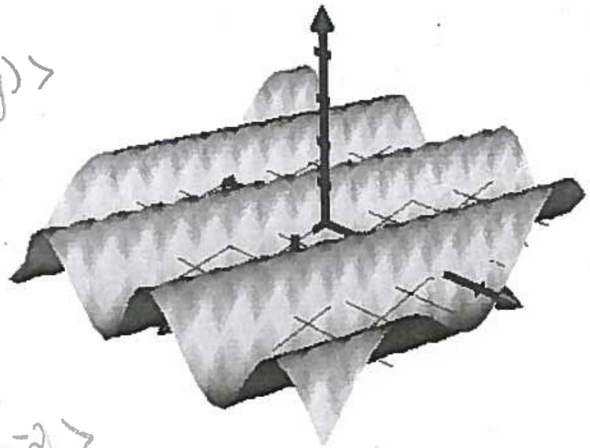
A. $z = \frac{x^2}{4} - y^2$ B. $z = |x| + |y| - 3$ C. $y^2 = x^2 + 2z^2$


B


A


C

7. Consider the function $f(x, y) = -\sin(x + 2y)$ for the following questions.



(a) [3] Find the gradient of f .

$\langle \underbrace{-\cos(x+2y)}_{+1}, \underbrace{-2\cos(x+2y)}_{+1} \rangle$

start +.5

notation +.5

(b) [1] Evaluate the gradient at the point $(0, 0)$.

$\langle -\cos(0), -\cos(0) \rangle = \langle -1, -2 \rangle$

notation +.5 plug in +.5

(c) [2] Interpret your answer in (b) graphically and consider referencing the graph of f shown to the right.

+.5 start.
+1 correct
1.5 sense.

The function f is increasing in the z coord. the fastest if you move in the direction of $\langle -1, -2 \rangle$. This is consistent with the picture since at (0,0), the fastest elevation change up, is moving back into the page to move up that ridge (in the $\langle -1, -2 \rangle$ direction).

(d) [3] Find the linear approximation of f at the point $(0, 0)$.

ie find the plane +.5

Note from (a) & (b)

$\left. \begin{array}{l} f_x(0,0) = -1. \\ f_y(0,0) = -2 \end{array} \right\}$ +1

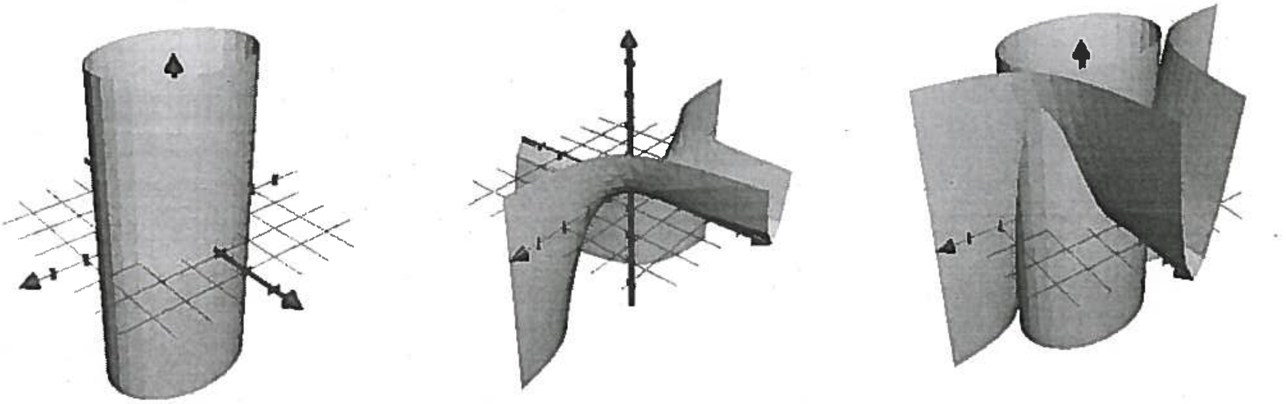
Thus we only need a point on the plane and (0,0,0) works so

+.5 eq of plane

$-1(x-0) + -2(y-0) = z - 0$ +1 plug in values

or $z = -x - 2y$.

8. Both the cylinder described by $6 = x^2 + 2y^2$ and the surface described by $z = x^2y$, along with their intersections are shown below.



(a) [6] Use Calculus methods to to find the (x, y, z) coordinates in $\mathbb{R}^3$ to find the points on both surfaces with the maximum z value.

reduced to calc 1 method:

boundary: $f(x,y) = x^2y$ where $6 = x^2+2y^2 \Rightarrow x^2 = 6-2y^2$ } sub (+1)

(+.5) start

(+.5) method Understand

(+1) plan

$\Rightarrow f(x,y) = f(y) = (6-2y^2)y = 6y-2y^3$

Critical points: $f'(y) = 6-6y^2$ (+.5) $\Rightarrow 0 = 6-6y^2$ (+.5) $\Rightarrow y = 1$ or -1 (+.5) (+.5)

$\Rightarrow x^2 = 6-2(1)^2 \Rightarrow x = 2$ or -2 so $(2,1)$ $(-2,1)$

$+ x^2 = 6-2(-1)^2 \Rightarrow x = 2$ or -2 $(2,-1)$ $(-2,-1)$ are CP

Determine is min/max: $6-6(-2)^2$ $6-60^2$ $6-6(2)^2$

$-$ -1 $+$ 1 $-$

↘ Min ↗ MAX ↘

(+1) some justification given (need not be this method)

So max values at

$(-2, 1, (-2)^2 \cdot 1) = (-2,1,4)$ and $(2, 1, 2^2 \cdot 1) = (2,2,4)$

(+.5) start

(+.5) method

(+1) plan ie Lagrange's

Using Lagrange multipliers: $f(x,y) = x^2y$ and $g(x,y) = x^2+2y^2 = 6$

$\nabla f(x,y) = \lambda \nabla g(x,y)$ (+.5)

$\langle 2xy, x^2 \rangle = \lambda \langle 2x, 4y \rangle$ (+.5) (+1)

$\Rightarrow \begin{cases} 2xy = 2\lambda x \\ x^2 = 4\lambda y \\ x^2+2y^2 = 6 \end{cases}$

if $x \neq 0$ $\begin{cases} y = \lambda \\ x^2 = 4\lambda y \\ x^2+2y^2 = 6 \end{cases}$ $\Rightarrow \begin{cases} x^2 = 4\lambda^2 \\ x^2+2\lambda^2 = 6 \end{cases}$

solve system of eq (+.5)

$\Rightarrow \lambda = 1$ or -1

$y = 1$ or -1

$x = \{2, -2$ or $x = \{2, -2$

so max/min at

$(2,1)$ $(2,-1)$

$(-2,1)$ $(-2,-1)$

justification (+1) looking at graph max at $(2,1)$ + $(-2,1)$

so $(2,1,4)$ + $(-2,1,4)$