Key

Exam 1 **TMath 126** Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) If $\vec{u}\cdot\vec{v}=0$, then $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$.

   false let $\vec{u}=\langle 0,0,1\rangle$ and $\vec{v}=\langle 1,0,0\rangle$. Then $\vec{u}\cdot\vec{v}=0\cdot1+0\cdot0+1\cdot0=0$ but $\vec{u}\neq\vec{0}$ and $\vec{v}\neq\vec{0}$

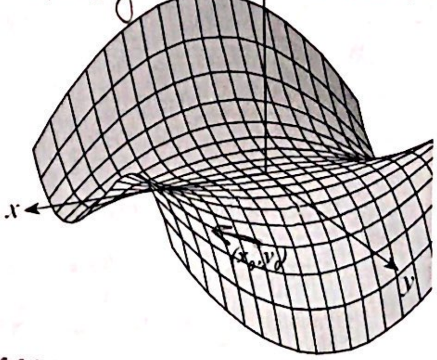
   (b) $(\vec{u}\times\vec{w})\cdot\vec{w}=0$

   True $\vec{u}\times\vec{w}$ produces a vector $\perp$ to both $\vec{u}$ and $\vec{w}$. Since $\vec{u}\times\vec{w}$ is $\perp$ to $\vec{w}$, the dot product of the 2 is $(\vec{u}\times\vec{w})\cdot\vec{w}=\|\vec{u}\times\vec{w}\|\,\|\vec{w}\|\cos 90^\circ=0$

   (c) $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}=\dfrac{\vec{u}}{\|\vec{u}\|}\cdot\dfrac{\vec{v}}{\|\vec{v}\|}$. If $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$

   true $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}=\dfrac{1}{\|\vec{u}\|\|\vec{v}\|}(u_1v_1+u_2v_2+u_3v_3)=\dfrac{u_1}{\|\vec{u}\|}\cdot\dfrac{v_1}{\|\vec{v}\|}+\dots+\dfrac{u_3}{\|\vec{u}\|}\cdot\dfrac{v_3}{\|\vec{v}\|}=\dfrac{\vec{u}}{\|\vec{u}\|}\cdot\dfrac{\vec{v}}{\|\vec{v}\|}$

problem → (d) The line $(2+3t,-4t,5+t)$ where $t\in\mathbb{R}$ intersects (this got cut off... I can't see?
   the point $(-4,8,3)$.



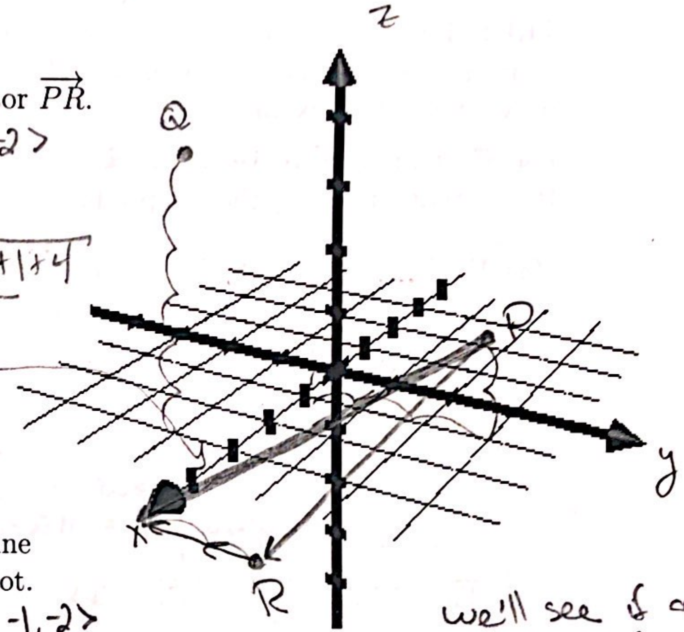
   (e) Consider the function g pictured to the right.
   $g_x(x_0,y_0)>0$.

   True as we move // to the x axis our z value increases

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the points: $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$. Also let $S(3,6,1.5)$ and $T(-9,-14,-12.5)$.

   (a) Plot the points P, Q, and R.



   (b) Find the components of the vector $\overrightarrow{PR}$.

   $\langle 5-1, 2-3, 0-2\rangle = \langle 4, -1, -2\rangle$

   (c) Find the length of $\overrightarrow{PR}$.

   $\sqrt{(5-1)^2+(2-3)^2+(0-2)^2} = \sqrt{16+1+4}$
   $= \sqrt{21}$

   (d) Draw the vector $\overrightarrow{PR} - 2\vec{j}$ and then write its components.

   $\langle 4,-1,-2\rangle - 2\langle 0,1,0\rangle$

   $\langle 4,-3,-2\rangle$

   (e) Use calculus methods to determine if $\triangle PQR$ is a right triangle or not.

   Non calculus method
   find $|PQ|$, $|PR|$ and $|QR|$.
   See if the lengths of edges satisfy $a^2+b^2=c^2$.
   Recall length of Δ
   $a^2+b^2=c^2 \Leftrightarrow$ right Δ

   $\overrightarrow{PR} = \langle 4,-1,-2\rangle$
   $\overrightarrow{PQ} = \langle 3-1, -1-3, 6-2\rangle = \langle 2,-4,4\rangle$
   $\overrightarrow{QR} = \langle 5-3, 2-1, 0-6\rangle = \langle 2,3,-6\rangle$

   we'll see if any of the pairs form a $90°$ with the dot product.

   $\overrightarrow{PR}\cdot\overrightarrow{PQ} = 4\cdot 2+4-8 \neq 0$
   $\overrightarrow{PR}\cdot\overrightarrow{QR} = 4\cdot 2-3+12 \neq 0$
   $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 4-12-24 \neq 0$

   no $90°$ angles (happen when dot prod $=0$) so not a right Δ.

   (f) Find the equation of the plane that passes through P, R, and Q.

   Looking for $\vec{n}\cdot(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle) = 0$
   need $\vec{n}$ to be $\perp$ to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
   the cross product will give $\perp$ vector

   $$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \vec{i}(8+4) - \vec{j}(-4-16) + \vec{k}(-2+16)$$
   $$= 12\vec{i} + 20\vec{j} + 14\vec{k}$$

   so $\langle 12,20,14\rangle\cdot(\langle x,y,z\rangle - \langle 3,-1,6\rangle) = 0$ works

   Note: there are lots of answers here!

   (g) Does the line that passes through S and T intersect the plane you found in part (a)? Justify yourself.

   $\overrightarrow{TS} = \langle 3+9, 6+14, 14\rangle$
   $= \langle 12, 20, 14\rangle$

   looks // to $\vec{n}$ found above!

   so yeah, the line $\overleftrightarrow{ST}$ will intersect the plane in (f)

3. [3] Consider the equation $2z = \frac{x^2}{2} - 2y^2$.

(a) Does the above equation describe a function of x and y? Why or why not?

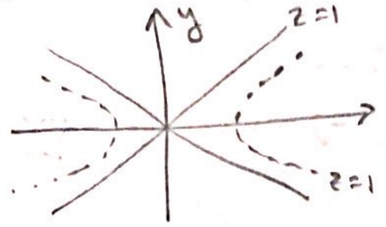
Sure, each (x,y) pair returns 1 z value

(b) Describe the contour curves of the graph of the equation above. That is, describe the intersection of the graph of the above equation with the planes $z = k$ where k is some constant.

$2K = \frac{x^2}{2} - 2y^2$

$\Rightarrow 4K = x^2 - 4y^2$

$\Rightarrow$ Hyperbolas?

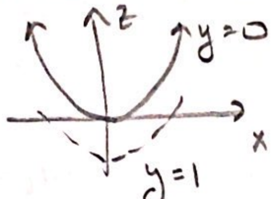


(c) Describe the intersection of the graph of $2z = \frac{x^2}{2} - 2y^2$ with planes parallel to the xz axis. That is, when $y = k$ for some constant k.

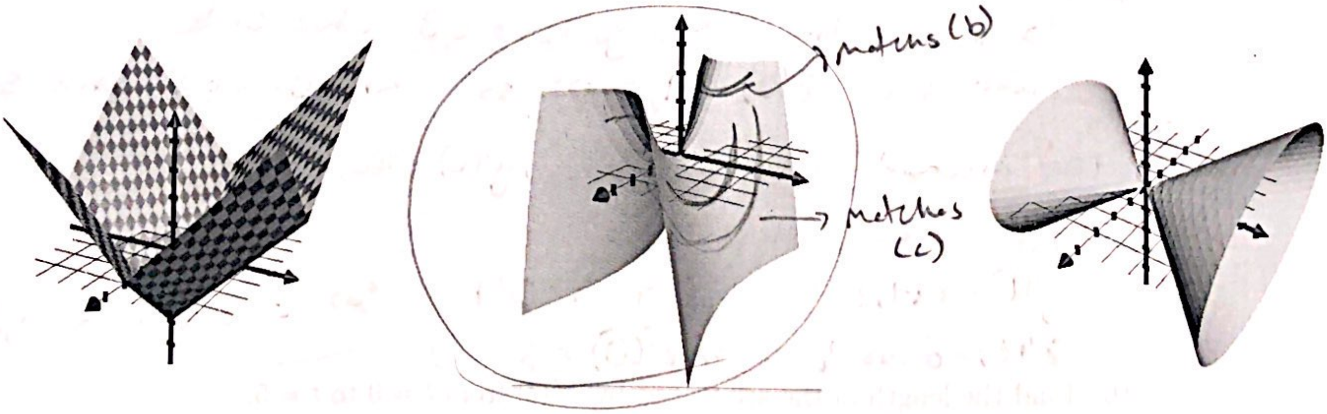
$2z = \frac{x^2}{2} - 2k^2$

$z = \frac{x^2}{4} - k^2$

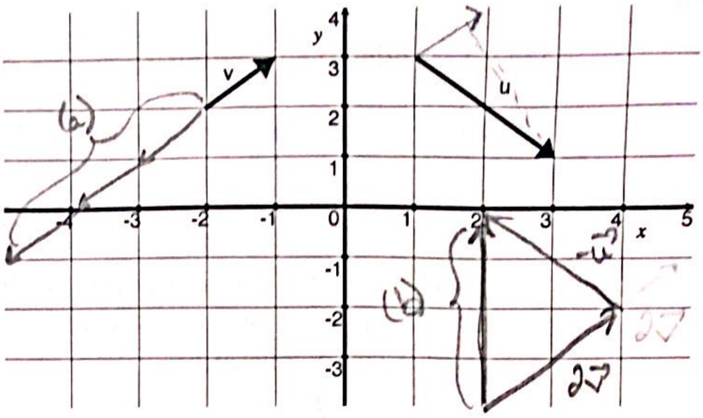
looks like parabolas



(d) Which (if any) of the following is a graph of the above function?



4. Consider the vector $\vec{v}$ and $\vec{u}$ shown to the right.



   (a) Draw the vector $-3\vec{v}$.

   (b) Draw the vector $2\vec{v} - \vec{u}$.

   (c) Find the projection of $\vec{u}$ onto $\vec{v}$.

... are these $\perp$?

$\langle 1,1\rangle \circ \langle 2,-2\rangle$

$= 2-2=0$

perpendicular ? !!

projection will be $\vec{0}$

OR $\quad \text{Proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

$= \dfrac{0}{\sqrt{2}^2}\langle 1,1\rangle = \vec{0}$

5. We define $\vec{r}(t)$ by: $x(t) = 1 + t^4$, $y(t) = te^{-t}$, and $z(t) = \sin(2t)$.

   (a) Find the line tangent to the curve $\vec{r}(t)$ when $t = 0$.

looking for a line $\langle x_0, y_0, z_0\rangle + s\vec{d}$ where $s \in \mathbb{R}$

when $t=0$ $\vec{r}(0) = \langle 1,0,0\rangle$ so $\langle 1,0,0\rangle + s\vec{d}$ where $s \in \mathbb{R}$

need directional vector so $(x'(0), y'(0), z'(0)) = \langle 0,1,2\rangle$

$x'(t) = 0 + 4t^3 \Rightarrow x'(0) = 0$

$y'(t) = t(-1)e^{-t} + e^{-t} \Rightarrow y'(0) = 0+1$

$z'(t) = 2\cos(2t) \Rightarrow z'(0) = 2$

so $\langle 1,0,0\rangle + s\langle 0,1,2\rangle$

   (b) Find the length of the arc traced by $\vec{r}(t)$ from $t = 0$ to $t = 5$.

$$\int_0^5 \sqrt{(4t^3)^2 + (e^{-t} - te^{-t})^2 + (2\cos(2t))^2}\, dt = 626$$

(technology)

6. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in feet) can be modeled by the function

$$T(x,t) = T_0 + T_1 e^{-\lambda x}\sin(\omega t - \lambda x)$$

where $\omega = 2\pi/360$ and λ is a positive constant.

(a) Find $\frac{\partial T}{\partial x}$. What is its physical significance?

(b) Find $\frac{\partial T}{\partial t}$. What is its physical significance?

(a) $\frac{\partial T}{\partial x} = 0 + (T_1 e^{-\lambda x})\frac{\partial}{\partial x}(\sin(\omega t - \lambda x)) + \frac{\partial}{\partial x}(T_1 e^{-\lambda x})\sin(\omega t - \lambda x)$

$= T_1 e^{-\lambda x}\cos(\omega t - \lambda x)\cdot(-\lambda) + T_1(-\lambda)e^{-\lambda x}\sin(\omega t - \lambda x)$

as depth (x) increases $\partial T/\partial x$ gives an approx change in temperature (T)

or the instantaneous rate of change in temperature (T) as depth (x) changes

(b) $\partial T/\partial t = 0 + T_1 e^{-\lambda x}\cos(\omega t - \lambda x)\cdot \omega$

$= T_1 e^{-\lambda x}\omega\cos(\omega t - \lambda x)$

as time (t) increases $\partial T/\partial t$ gives an approx change in temperature (T)