Key

Exam 2 **TMath 126** Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) Let f be a function of x and y. If $\nabla f(c,d) = (2,1)$, then the vector $\langle 2,1\rangle$ is tangent to the contour line of the surface of f at $(c,d,f(c,d))$.

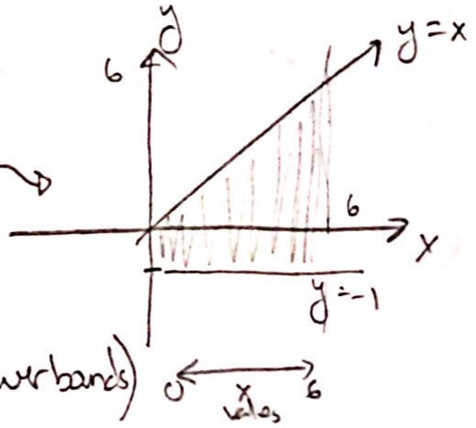
   False $\nabla f(c,d)$ points in the direction of steepest assent. The contour line would keep the "elevation"/z value constant. So $\nabla f(c,d)$ is $\perp$ to contour line

   (b) $\int_{-1}^{2}\int_0^6 x^2\sin(x-y)\,dx\,dy = \int_0^6\int_{-1}^{2} x^2\sin(x-y)\,dy\,dx$

   True. The function $x^2\sin(x-y)$ is continuous on the rectangle $[0,6]\times[-1,2]$ so we can use Fubini's Thm.
   (or we could compute each individually? ... would involve integration by parts on one side...)

   (c) $\int_{-1}^{x}\int_0^6 x^2\sin(x-y)\,dx\,dy = \int_0^6\int_{-1}^{x} x^2\sin(x-y)\,dy\,dx$

   False. The integral ↑ corresponds with the volume above the area →
   If we reversed the order of integration we'll have to split the computation (b/c 2 dif. lowerbounds)



$$\int_0^6\int_y^6 x^2\sin(x-y)\,dx\,dy + \int_{-1}^{0}\int_0^6 x^2\sin(x-y)\,dy\,dx$$

Note: This practice exam is light on Section 14.6 but I wanted to get *something* to you!!! Check out class activities, WebHW & WrittenHW from the section and add that here! (Pick something with pictures, I like pictures.)

2. You are given the following data of a function $g(x, y)$. Your boss wants you to approximate $g(.8, 1.4)$ and wants to be convinced you're doing something sophisticated. Find a linear approximation for your boss and explain your choices (there are many that you will make!).

x	y	$g(x, y)$	
0.55	1.2	27	1st
0.65	1.0	31	
0.65	1.1	29	
0.75	1.2	50	4th

We will use a 3D linear approximation to approximate $g(.8, 1.4)$.

The linear approx for 3D is of the form

$$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$$

Lets use $(.75, 1.2, 50)$ as this point has the x + y values closest to 0.8 and 1.4 respectively.

$$\Rightarrow \quad z - 50 = m_x(x - .75) + m_y(y - 1.2)$$

To find $m_x = \left.\frac{dz}{dx}\right|_{y=1.2} \approx \frac{\Delta g}{\Delta x} = \frac{50-27}{.75-.55} = \frac{23}{.2} = 115$

need to keep y constant so using 1st + 4th

To find $m_y = \left.\frac{dz}{dy}\right|_{x=.65} \approx \frac{\Delta g}{\Delta y} = \frac{31-29}{1.0-1.1} = \frac{2}{-.1} = -20$

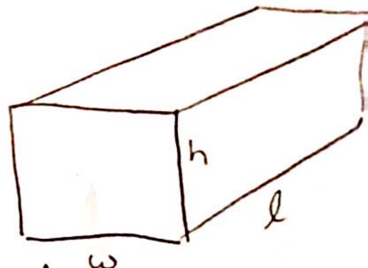
need to keep x constant so using 2nd + 3rd

So one approx is $z - 50 = 115(x - .75) + {}^{-}20(y - 1.2)$

$$\Rightarrow g(.8, 1.4) \approx 50 + 115(.8 - .75) - 20(1.4 - 1.2)$$

3. Find the maximum and minimum volumes of a rectangular box with the constraints that the surface area is 1500cm^2 and total edge length is 200cm.

V = (Maximize) volume = $l \cdot w \cdot h$



Constraints: $2lh + 2lw + 2hw = 1500$

$4l + 4w + 4h = 200$

Could do a series of substituting but I'll use Lagrange Multipliers

$\nabla V = \nabla(lwh) = \lambda \nabla(2lh + 2lw + 2hw) + \mu \nabla(4l + 4w + 4h)$

$$\begin{array}{l} (\text{wrt } l) \\ (\text{wrt } w) \\ (\text{wrt } h) \\ \\ \\ \end{array} \left\{ \begin{array}{l} wh = \lambda(2h + 2w) + \mu(4) \\ lh = \lambda(2l + 2h) + \mu(4) \\ lw = \lambda(2l + 2w) + \mu 4 \\ 2lh + 2lw + 2hw = 1500 \\ 4l + 4w + 4h = 200 \end{array} \right\}$$

5 equations / 5 unknowns

4. Common blood types are determined by three alleles, A, B, and O. If p is the percent of allele A in the population, q is the percent of allele B in the population and r is the percent of allele O in the population then the proportion of individuals with a mixed blood type (e.g. AB, AO or BO) is $P(p, q, r) = 2pq + 2pr + 2qr$. Find the maximal P value.

Maximize $P = 2pq + 2pr + 2qr$

note $p + q + r = 1$ ⟵ total population

↑ % of A, ↑ % of B, ↑ % of O

More Lagrange? Maybe I'll use substitution instead..

$p + q + r = 1 \Rightarrow p = 1 - q - r$

$\Rightarrow P = 2(1-q-r)q + 2(1-q-r)r + 2qr$

$= 2q - 2q^2 - 2rq + 2r - \cancel{2qr} - 2r^2 + \cancel{2qr}$

$= -2q^2 + 2q - 2rq + 2r - 2r^2$

To optimize function of 2 variables

$\frac{\partial P}{\partial q} = 0 \Rightarrow -4q + 2 - 2r = 0$

AND

$\frac{\partial P}{\partial r} = 0 \Rightarrow -2q + 2 - 4r = 0$

Finding Critical Points $\left\{ \begin{array}{l} -4q + 2 - 2r = 0 \\ -2q + 2 - 4r = 0 \end{array} \right\} \begin{array}{l} (1) \\ (2) \end{array}$

Use (1) to solve for $r \Rightarrow \frac{-4q + 2}{2} = r$

$r = -2q + 1$

Sub into (2) $\quad -2q + 2 - 4(-2q + 1) = 0$

$-2q + 2 + 8q - 4 = 0$

$6q = 2 \Rightarrow q = \frac{1}{3}$

Since $r = -2(q) + 1 = -2(\frac{1}{3}) + 1 = -\frac{2}{3} + 1 = \frac{1}{3}$

Since $p = 1 - \frac{1}{3} - \frac{1}{3} \Rightarrow p = \frac{1}{3}$

Critical Point: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ Max value?

Use 2nd derivative Test

$\frac{\partial P^2}{\partial q^2} = -4 \quad \frac{\partial P^2}{\partial r^2} = -4 \quad \frac{\partial P^2}{\partial q \partial r} = -2$

$D = (-4)(-4) - (-2)^2 = 16 - 4 > 0$

$\frac{\partial P^2}{\partial q^2} = -4 < 0 \Rightarrow$ max value @

$P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 2 \cdot \frac{1}{9} + 2\frac{1}{9} + 2\frac{1}{9} = \frac{6}{9} = \boxed{\frac{2}{3}}$

5. Consider the double integral

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2 x}\, dx dy$$

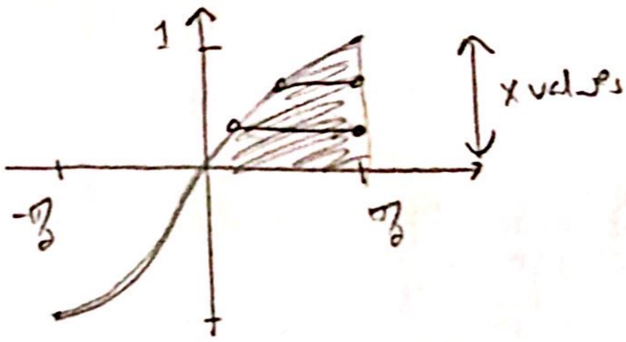
(a) Sketch the region in the xy-plane where the integral is taken over.

$x = \arcsin(y)$

$\Leftrightarrow$

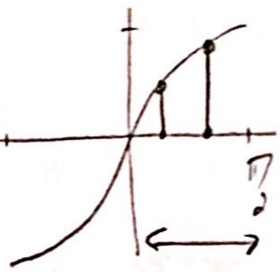
$\sin(x) = y$

where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



(b) Switch the order of integration.

$$\int_0^{\pi/2}\int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2(x)}\,dy\,dx$$



(c) Compute the double integral.

The initial integral looks messy... maybe pythagoras could make nicer?

$\sin^2 x + \cos^2 x = 1$ ... nope, need subtraction... uh trig sub?

Yuck, .. I'm going to the one we came up with in (b)

$$\int_0^{\pi/2}\int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2(x)}\,dy\,dx = \int_0^{\pi/2} \cos(x)\sqrt{1+\cos^2(x)}\; y\Big]_0^{\sin x} dx$$

$$= \int_0^{\pi/2} \sin(x)\cos(x)\sqrt{1+\cos^2(x)}\,dx$$

ug... I'm back.. how about trig ID?

how about $u = 1+\cos^2(x)$

$du = -2\cos(x)\sin(x)dx$

$\frac{1}{-2\cos(x)\sin(x)}du = dx$

$$\int \cancel{\sin(x)}\cancel{\cos(x)}\sqrt{u}\,\frac{1}{-2\cancel{\cos(x)}\cancel{\sin(x)}}du$$

$$= \frac{1}{-2}\int u^{1/2}du = -\frac{1}{\cancel{2}}\frac{\cancel{2}}{3}u^{3/2} + C$$ ☺

$$-\frac{1}{3}\left(1+\cos^2(x)\right)^{3/2}\Big]_0^{\pi/2} = -\frac{1}{3}\left(1+\cos^2\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{1}{3}\left(1+\cos^2(0)\right) = -\frac{1}{3} + \frac{1}{3}2^{3/2} = \frac{-1+\sqrt{8}}{3}$$