Key

EXAM 2 — TMath 125 — Spring 2014

1. [3] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

   T **(F)** $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$ for real, non zero numbers a and b. $\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab}$

   T **(F)** If g is continuous and $\int_{-3}^{7} \frac{1}{2} g(x)\,dx = 4$, we can not evaluate $\int_0^5 g(2x-3)\,dx$.

   $u = 2x-3$
   $du = 2dx$
   $\frac{1}{2}du = dx$

   $\int_0^5 g(2x-3)\,dx = \int_{2\cdot 0-3}^{2\cdot 5-3} \frac{1}{2} g(u)\,du = \int_{-3}^{8} \frac{1}{2} g(u)\,du = 4$

   (T) F Calculus 2 & precalculus methods can integrate any rational function.

   yay Fundamental Theorem of Algebra!

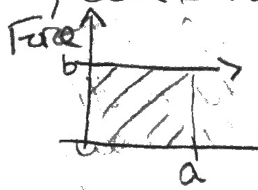
Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [4] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.

   (a) (Word Wks #2) If you are given the graph of a force as a function with respect to distance below, explain to someone who only took the first quarter of Calculus how to graph the total work as a function of distance.

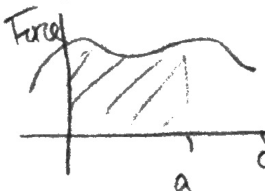
   (b) (Word Wks #4) Explain the Mean Value Theorem for Integrals to someone who only took the first quarter of Calculus.

a) Work is computed as Force·distance. (+1)



So if the force is a constant b & we travel a meters, the work is a·b (+1)
or the area trapped below the force function from 0 to a.

When Force varies as a function of distance it's harder to compute but still the area trapped below the force function from 0 to a



appropriate audience (.5)
sense (.5)
correct (.5)
on topic (.5)

b) The Mean Value Theorem is talking about 'the average value of a function'.
Recall that we can find the average value of n numbers by adding the #s up & dividing by n.
We do the same thing in Calc. We add up the values ($\int_a^b f(x)\,dx$) and divide by the "# of numbers" or $b-a$ so $\frac{1}{b-a} \cdot \int_a^b f(x)\,dx$. (+1)
The cool thing is that here we know somewhere between a & b (+1) the function actually evaluates to the average. that is there exists a c between a & b so that
$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$.

3. [12] For each of the following outline the method(s) you would use to find the general antiderivative. Include in the descriptions which substitutions you would make and the fall out that would occur. Essentially, give the same level of detail as was given on the practice exam key.

sense (+.5)
to integration (+.5)

[3] (sorry-no meta-data for these)

$$\int \cos^3(\theta)\sin^3(\theta)\,d\theta$$

We'll use the substitution $u = \sin\theta$. Reserve one $\cos\theta$ factor for the du (+1)

and convert the other $\cos^2\theta$ factor into sines with Pythagoras' $\sin^2\theta + \cos^2\theta = 1$. (+1)

After that you'll have a polynomial in u that you can integrate using the backwards power rule.

Note: this is not the only way!

[3] (sorry-no meta-data for these)

$$\int \frac{2x^3+2x+1}{x^2+1}\,dx$$

sense (+.5)
to integration (+.5)

(+.5) Use long division to write the "improper" rational function as a polynomial + a "proper" rational function

(+.5) The polynomial can be integrated with the backwards power rule.

(+1) The 'proper' rational function can be integrated with substitution. If there is an x in the numerator we'll use ln + if there is a constant we'll use arctan

[3] (sorry-no meta-data for these)

$$\int \frac{t^3}{\sqrt{t^2+49}}\,dt$$

sense (+.5)
to integration (+.5)

We'll use substitution + let $u = t^2+49$ + $du = 2t\,dt$. Note $t^2 = u-49$ so we can rewrite the numerator to (+1)

$$\int \frac{(u-49)}{u^{1/2}}\frac{1}{2}\,du = \frac{1}{2}\int \frac{u}{u^{1/2}} - \frac{49}{u^{1/2}}\,du$$

(+1) We can then use exponent rules to clean up the integrand + use the backwards power rule.

note: this problem could be done with trigonometric substitution

[3] (sorry-no meta-data for these)

$$\int \frac{\ln y}{\sqrt{y}}\,dy$$

sense (+.5)
to integration (+.5)

$u = \ln y \quad v = 2y^{1/2}$

$du = \frac{1}{y}dy \quad dv = y^{-1/2}\,dy$

(+1.5) Use integration by parts with the above u + dv to get

$$(\ln y)(2y^{1/2}) - \int 2y^{1/2}\cdot\frac{1}{y}\,dy$$

(+.5) We can then use exponent rules to clean up the integrand + use the backwards power rule

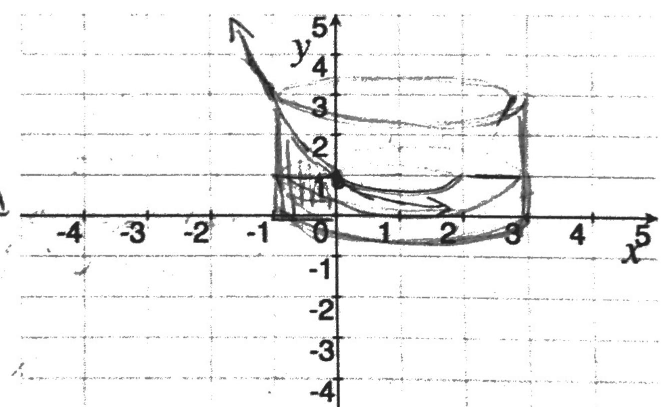
4. [4] Identify one integral from the previous page and find the general antiderivative.

See attached paper.

5. [5] (§7.1 #63) Consider the region under the curve $y = e^{-x}$, above $y = 0$, and from $-1 \leq x \leq 0$.

(a) [1] Carefully draw the region described.

y=e^x but horizontally flipped



(b) [4] Set up the integral (but do *not* integrate) that is used to find the volume of the solid resulting from rotating the above region about the line $x = 1$.

Big cylinder - hole

notice the hole is another cylinder when $y<1$ with hight 1 & radius 1

$\pi 2^2 \cdot e - (\quad + \quad)$

note $y = e^{-x} \Rightarrow \ln y = -x$
$\Rightarrow -\ln y = x$

$4e\pi - (\pi 1^2 \cdot 1 + \text{Sum } \pi r^2 \Delta y)$

$4e\pi - \pi - \int_1^e \pi(1+x)^2 dy = 4e\pi - \pi - \int_1^e \pi(1-\ln y)^2 dy$

$$\int \cos^3(\theta)\sin^3(\theta)\,d\theta = \int \cos^2(\theta)\sin^3(\theta)\cos(\theta)\,d\theta$$

$u = \sin(\theta)$

(+.5) $du = \cos(\theta)\,d\theta$

recall $\sin^2(\theta)+\cos^2(\theta)=1$

$\Rightarrow \cos^2(\theta) = 1-\sin^2(\theta)$

$$= \int (1-\sin^2(\theta))\sin^3\theta\cos(\theta)\,d\theta$$ by pythagoras

$$= \int (1-u^2)u^3\,du$$ by substitution

$$= \int u^3-u^5\,du$$ distributing

(+1) $= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$ (+.5)

$$= \frac{1}{4}\sin^4(\theta) - \frac{1}{6}\sin^6(\theta) + C$$

sub (+.5)
distribut (+.5)
algebra (+.5)
answer in θ (+.5)

Ck: $\frac{d}{d\theta}\left(\frac{1}{4}\sin^4(\theta) - \frac{1}{6}\sin^6(\theta) + c\right)$

$= \frac{1}{\cancel{4}}\cdot\cancel{4}\sin^3(\theta)\cos(\theta) - \frac{1}{\cancel{6}}\cancel{6}\sin^5(\theta)\cos(\theta) + 0$

$= \sin^3(\theta)\cos(\theta) - \sin^5(\theta)\cos\theta$

$= \sin^3(\theta)\cos(\theta)\,[1-\sin^2(\theta)]$

$= \sin^3(\theta)\cos(\theta)\cos^2(\theta)$ ✓

$$\int \frac{2x^3+2x+1}{x^2+1}\,dx = \int 2x + \frac{1}{x^2+1}\,dx = \underbrace{x^2}_{+.5} + \underbrace{\arctan(x)}_{+.5} + \underbrace{C}_{+.5}$$

$$\begin{array}{r} 2x \quad R1 \\ x^2+1 \,\overline{)\, 2x^3+0x^2+2x+1} \\ -(2x^3 \quad +2x) \\ \hline 1 \end{array}$$

Ck: $2x + \frac{1}{x^2+1}$

$= \frac{2x(x^2+1)+1}{x^2+1}$ ✓

long ÷ (+.5)
distribut (+1)
algebra (+1)

Ck: $\frac{d}{dx}\left(x^2 + \arctan(x) + c\right)$

$= 2x + \frac{1}{x^2+1} + 0$

$= \frac{2x(x^2+1)+1}{x^2+1} = \frac{2x^3+2x+1}{x^2+1}$ ✓

$$\int \frac{t^3}{\sqrt{t^2+49}}\,dt = \int \frac{t^2}{\sqrt{u}}\,t\,dt$$

$u = t^2+49$

$u-49 = t^2$

(+.5) $du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$= \int \frac{u-49}{u^{1/2}}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int \frac{u}{u^{1/2}} - \frac{49}{u^{1/2}}\,du$$

$$= \frac{1}{2}\int u^{1/2} - 49u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 49\cdot 2u^{1/2}\right] + C$$

(+.5) (+.5) (+.5)

$$= \frac{1}{3}u^{3/2} - 49u^{1/2} + C$$

$$= \frac{1}{3}(t^2+49)^{3/2} - 49(t^2+49)^{1/2} + C$$

sub (+.5)
did right (+.5)
algebra (+.5)
answer in t (+.5)

CK: $\frac{d}{dt}\left(\frac{1}{3}(t^2+49)^{3/2} - 49(t^2+49)^{1/2} + C\right)$

$$= \frac{1}{3}\cdot\frac{3}{2}\sqrt{t^2+49}\,2t - 49\cdot\frac{1}{2}\frac{1}{\sqrt{t^2+49}}\cdot 2t + 0$$

$$= \left[(t^3+49t) - 49t\right]/\sqrt{t^2+49} \quad ✓$$

$$\int \frac{t^3}{\sqrt{t^2+49}}\,dt = \int \frac{7^3\tan^3(\theta)}{\sqrt{49\tan^2(\theta)+49}}\cdot 7\sec^2\theta\,d\theta$$

$\tan^2(\theta)+1 = \sec^2(\theta)$

$t = 7\tan(\theta)$

$dt = 7\sec^2(\theta)\,d\theta$

$$= 7^3\int \frac{\tan^3(\theta)}{\cancel{7\sec(\theta)}}\,\cancel{7}\sec^{\cancel{2}}\theta\,d\theta$$

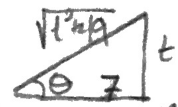
where $\sec(\theta) > 0$

$u = \sec(\theta)$

$du = \tan(\theta)\sec(\theta)$

$$= 7^3\int \tan^2(\theta)\,\tan\theta\sec\theta\,d\theta$$

OR



$\tan\theta = t/7$

$\sec\theta = \frac{\sqrt{t^2+49}}{7}$

$$= 7^3\int (\sec^2\theta - 1)\,du$$

$$= 7^3\int u^2 - 1\,du$$

$$= 7^3\left[\frac{1}{3}u^3 - u\right] + c$$

$$= 7^3\left[\frac{1}{3}\sec^3(\theta) - \sec(\theta)\right] + c$$

$$= \frac{7^3}{3}\left(\frac{\sqrt{t^2+49}}{7}\right)^3 - 7^{\cancel{3}}\left(\frac{\sqrt{t^2+49}}{7}\right) + c$$

yuck

$$= \frac{1}{3}(t^2+49)^{3/2} - 49(t^2+49)^{1/2} + c$$

$$\int \frac{\ln y}{\sqrt{y}}\,dy = \int (\ln y)\,y^{-1/2}\,dy = uv - \int v\,du$$

(+.5)

$$= (\ln y)(2y^{1/2}) - \int 2y^{1/2}\,\frac{1}{y}\,dy \quad (*1)$$

$u = \ln y$

$du = \frac{1}{y}dy$

(+.5) $v = 2y^{1/2}$

$dv = y^{-1/2}dy$

$$= (\ln y)(2y^{1/2}) - 2\int \frac{1}{y^{1/2}}\,dy$$

$$= (\ln y)(2y^{1/2}) - 2\int y^{-1/2}\,dy$$

$$= 2\sqrt{y}\,\ln y - 2\cdot 2\,y^{1/2} + c$$

(+.5)

$$= 2\sqrt{y}\,\ln y - 4y^{1/2} + c$$

(+.5)

IP (+.5)

algebra (+.5)

CK: $\frac{d}{dy}\left(2\sqrt{y}\,\ln y - 4y^{1/2} + c\right) = 2\sqrt{y}\,\frac{1}{y} + 2y^{-1/2}\cdot\frac{1}{2}\ln y - 4\cdot\frac{1}{2}y^{-1/2} + 0$

$$= \frac{\cancel{2}}{\cancel{\sqrt{y}}} + \frac{\ln y}{\sqrt{y}} - \frac{\cancel{2}}{\cancel{\sqrt{y}}} = \frac{\ln y}{\sqrt{y}} \quad ✓$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

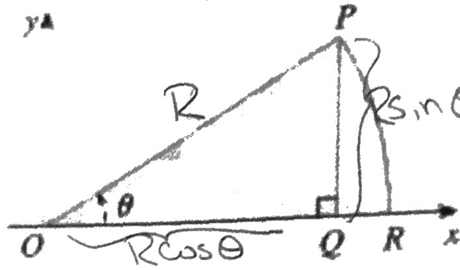
$\cos(2\theta) = \cos^2\theta - (1-\cos^2\theta)$

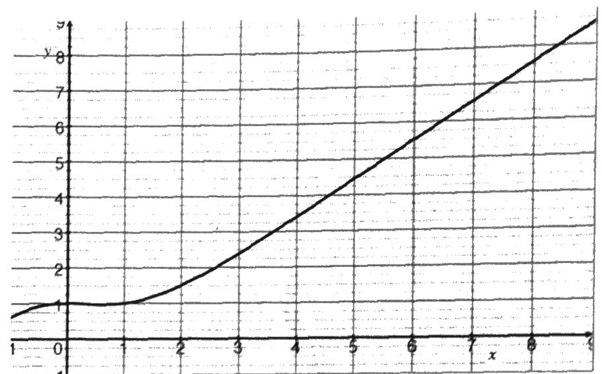
$\cos(2\theta) = (1-\sin^2\theta) - \sin^2\theta$

6. [7] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.

   (a) (Word Wks #1) Prove the formula $A = \frac{1}{2}r^2\theta$ for the area of a sector of a circle with radius r and central angle θ (pictured to the right).

   (b) A particle is moving along in a straight line and has the velocity of $v(t) = \frac{t^3+4}{t^2+4}$ at time t. (pictured to the right) Find a function that describes the net change in position at time t.





a) The area of the sector is

triangle + wedge

Area of triangle $= \frac{1}{2} \cdot \text{base} \cdot \text{height}$

$= \frac{1}{2} \cdot R\cos\theta \cdot R\sin\theta$

Area of the wedge $= \int_{R\cos\theta}^{R} \sqrt{R^2-x^2}\,dx$

(b/c the circle is $x^2+y^2=R^2$)

$\int_{R\cos\theta}^{R} \sqrt{R^2-x^2}\,dx$ let $x = R\cos\varphi \cdot d\varphi$, $dx = -R\sin\varphi\, d\varphi$

$-\int \sqrt{R^2-R^2\cos^2\varphi}\; R\sin\varphi\, d\varphi$

$= -\int R\sqrt{1-\cos^2\varphi}\; R\sin\varphi\, d\varphi$

$= -R^2\int \sin\varphi \sin\varphi\, d\varphi$ as long as $\cos\varphi > 0$

$= -R^2\int \sin^2\varphi\, d\varphi = -R^2\int \frac{1}{2}(1-\cos(2\varphi))\,d\varphi$

$= -\frac{R^2}{2}\left[\varphi + \frac{1}{2}\sin(2\varphi)\right] + C$

$= -\frac{R^2}{2}\left[\varphi - \frac{1}{2}\cancel{2}\sin\varphi\cos\varphi\right] + C$

$-\frac{R^2}{2}\left[\arccos\left(\frac{x}{R}\right) + \frac{x}{R}\frac{\sqrt{R^2-x^2}}{R}\right] + C$

So Area of wedge

$\int_{R\cos\theta}^{R} \sqrt{R^2-x^2}\,dx = -\frac{R^2}{2}\left[\arccos\left(\frac{R}{R}\right) + \cancel{\frac{R}{R}\frac{\sqrt{R^2-R^2}}{R}}\right]^{0}$

$+ \frac{R^2}{2}\left[\arccos\left(\frac{R\cos\theta}{R}\right) + \frac{R\cos\theta\sqrt{R^2-R^2\cos^2\theta}}{R^2}\right]$

$= \frac{R^2}{2}\left(\arccos(\cos\theta) + \frac{R\cos\theta\sin\theta}{R}\right)$

$= \frac{R^2}{2}(\theta + \cos\theta\sin\theta)$

$= \frac{R^2\theta}{2} - \frac{R^2\cos\theta\sin\theta}{2}$

So triangle + wedge

$\cancel{\frac{1}{2}R^2\cos\theta\sin\theta} + \frac{R^2\theta}{2} - \cancel{\frac{R^2\cos\theta\sin\theta}{2}}$

$= \frac{R^2\theta}{2}$

b) Recall

$\frac{d}{dt}\left(\begin{array}{c}\text{position} \\ \text{velocity}\end{array}\right)\int$

$\therefore$ to find the ~~the~~ net position $p(t)$ we need $\int_0^t v(x)\,dx$.

$p(t) = \int v(x)\,dx = \int_0^t \frac{x^3+4}{x^2+4}\,dx$

$= \int_0^t x + \frac{-4x+4}{x^2+4}\,dx$

$= \int_0^t x\,dx + \int_0^t \frac{-4x}{x^2+4} + \frac{4}{x^2+4}\,dx$

$= \frac{1}{2}x^2\Big]_0^t - 4\int_0^t \frac{x}{x^2+4}\,dx + 4\int_0^t \frac{1}{x^2+4}\,dx$

$= \frac{1}{2}t^2 - 4\cdot\int_4^{t^2+4} \frac{1}{2}\,\frac{1}{u}\,du + 4\int_0^t \frac{1}{x^2+4}\,dx$

$= \frac{1}{2}t^2 - 2\ln|u|\Big]_4^{t^2+4} + \cancel{4}\int_0^t \frac{1}{\cancel{4}\left(\frac{x^2}{4}+1\right)}\,dx$

$= \frac{1}{2}t^2 - \left[2\ln(t^2+4) - 2\ln 4\right] + 2\int_0^{t/2} \frac{1}{w^2+1}\,dw$

$= \frac{1}{2}t^2 - 2\ln(t^2+4) + 2\ln 4 + 2\arctan(w)\Big]_0^{t/2}$

$= \frac{1}{2}t^2 - 2\ln(t^2+4) + 2\ln 4 + 2\arctan\left(\frac{t}{2}\right) - 2\arctan(0)$

$= \frac{1}{2}t^2 - 2\ln(t^2+4) + 2\ln 4 + 2\arctan\left(\frac{t}{2}\right)$

$$\begin{array}{r} x \quad R\ -4x+4 \\ x^2+4\ \overline{)\ x^3+0x^2+0x+4} \\ -(x^3 \qquad\quad +4x) \\ \hline -4x+4 \end{array}$$

Ck: $x + \frac{-4x+4}{x^2+4}$

$\frac{x(x^2+4) - 4x + 4}{x^2+4}$

$\frac{x^3 + \cancel{4x} - \cancel{4x} + 4}{x^2+4}$ ✓

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$w = \frac{x}{2}$

$dw = \frac{1}{2}dx$

$2dw = dx$