ex $\int \cos^2 x \sin^2 x \, dx$

$\overset{(\star)}{=} \int \frac{1}{2}[\cos(2x)+1] \sin^2 x \, dx$

$= \frac{1}{2}\int (\cos(2x)+1) \sin^2 x \, dx$

$\overset{(☺)}{=} \frac{1}{2}\int (\cos(2x)+1) \frac{1}{2}[1-\cos(2x)] \, dx$

$= \frac{1}{4}\int (\cos(2x)+1)(1-\cos(2x)) \, dx$

$= \frac{1}{4}\int \cancel{\cos(2x)} - \cos^2(2x) + 1 - \cancel{\cos(2x)} \, dx$

$= \frac{1}{4}\int 1 - \cos^2(\boxed{2x}) \, dx.$

$\overset{(\star)}{=} \frac{1}{4}\int 1 - \frac{1}{2}[\cos(2\boxed{2x})+1] \, dx$

$= \frac{1}{4}\int 1 - \frac{1}{2}[\cos(4x)+1] \, dx$

$= \frac{1}{4}\int 1 - \frac{1}{2}\cos(4x) - \frac{1}{2} \, dx$

$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos 4x \, dx$

$= \frac{1}{8}\int 1 - \cos 4x \, dx$

$u = 4x$

$du = 4dx \Rightarrow \frac{1}{4} du = dx$

$= \frac{1}{8}\left[\int 1 dx - \int \cos 4x dx\right]$

$= \frac{1}{8}\left[x - \int (\cos u) \frac{1}{4} du\right]$

$= \frac{1}{8}\left[x - \frac{1}{4}\int \cos u \, du\right]$

$= \frac{1}{8}\left[x - \frac{1}{4}\sin u\right] + C$

$= \frac{1}{8}\left[x - \frac{1}{4}\sin(4x)\right] + C = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C$

identities:

$\sin^2 x + \cos^2 x = 1$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin(x+y) = \cos x \sin y + \cos y \sin x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$\cos^2 x = \cos 2x + \sin^2 x$

$= \cos(2x) + (1-\cos^2 x)$

$\Rightarrow 2\cos^2 x = \cos(2x)+1$

$\star \Rightarrow \cos^2 x = \frac{1}{2}[\cos(2x)+1]$

$\sin^2 x = \cos^2 x - \cos 2x$

$= (1-\sin^2 x) - \cos 2x$

$\Rightarrow 2\sin^2 x = 1 - \cos 2x$

☺ $\Rightarrow \sin^2 x = \frac{1}{2}[1-\cos(2x)]$

$\star$ $\cos^2 \square = \frac{1}{2}[\cos(2\square)+1]$

2. Recall the Pythagorean Theorem (the trigonometric version of $a^2 + b^2 = c^2$)

$$\sin^2(x) + \cos^2(x) = 1.$$

(a) Use the above theorem to write down a relationship between $\tan(x)$ & $\sec(x)$.

$\frac{\sin^2(x)+\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$

(b) Use the above theorem to write down a relationship between $\cot(x)$ and $\csc(x)$.

$\frac{\sin^2 x+\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} \Rightarrow 1+\cot^2 x = \csc^2 x$

3. Consider the strategy we developed to integrate expressions with sines and cosines. Try and develop a parallel strategy when working the following examples:

$\int \tan^6(x)\sec^4(x)\,dy = \int \tan^6(x)\sec^2 x \sec^2 x\,dx$

$u = \tan(x)$
$du = \sec^2 x\,dx$

$= \int u^6 \sec^2 x\,du = \int u^6(\tan^2 x+1)\,du$

$= \int u^6(u^2+1)\,du$

$= \int u^8 + u^6\,du = \frac{1}{9}u^9 + \frac{1}{7}u^7 + c$

$= \frac{1}{9}\tan^9(x) + \frac{1}{7}\tan^7(x) + c$

$\int \tan(x)\sec^4(x)\,dy = \int \sec^3 x \sec x \tan x\,dx$

$u = \sec x$
$du = \sec x \tan x\,dx$

$= \int u^3\,du = \frac{1}{4}u^4 + c$

$= \frac{1}{4}\sec^4 x + c$

$\int \tan^4(x)\,dx = \int \tan^2 x \tan^2 x\,dx$

$u = \tan x$
$du = \sec^2 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$

$= \int \tan^2 x \sec^2 x - \tan^2 x\,dx$

$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$= \int u^2\,du - \int (\sec^2 x - 1)\,dx$

$= \frac{1}{3}u^3 - \int \sec^2 x - 1\,dx$

$= \frac{1}{3}\tan^3 x - (\tan x - x) + c$

$= \frac{1}{3}\tan^3 x - \tan x + x + c$

note: if the power on tan is odd - we can use it in du if $u = \sec x$

4. Record your strategy by finishing the following sentences: Given $\int \tan^m(x)\sec^n(x)\,dx$,

(a) if n is even... we can let $u = \tan x$ and use $du = \sec^2 x\,dx$. If we have any left over (even powers of) sec x we can use pyg

(b) if m is odd... we can set $u = \sec x$ and then use one $\tan(x)$ for the $du = \sec(x)\tan(x)\,dx$
If we have any left over (even powers of) tan we can use Pythagorus to switch them into $(\sec^2 x - 1)$'s.