FINAL

Math 252

PRACTICE

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let a and b be constants with $a \leq b$ and $f(x)$ and $g(x)$ be continuous functions on $[a, b]$.

T F We can differentiate any rudimentary collection of functions with calculus 1 methods.

T F We can integrate any rudimentary collection of functions with calculus 2 methods.

T F $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

T F $\displaystyle\int_a^b f(x)g(x)\,dx = \int_a^b f(x)\,dx * g(x) + f(x) * \int_a^b g(x)\,dx$ Product rule is for differentiation not integration

T F If f is continuous, then $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx.$ $\displaystyle\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_{a}^{\infty} f(x)dx$

T F If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty f(x)+g(x)\,dx$ is convergent.

$\displaystyle\int_a^\infty f(x)+g(x)dx = \lim_{t\to\infty}\int_a^t f(x)+g(x)dx = \lim_{t\to\infty}\left[\int_a^t f(x)dx + \int_a^t g(x)dx\right] = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

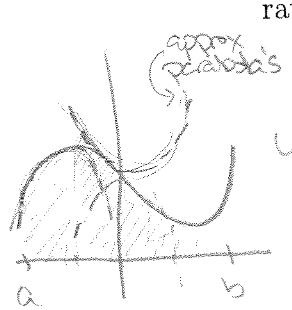
2. *Carefully* write down the first Fundamental Theorem of Calculus.

If f is continuous and $F(x) = \int_a^x f(t)dt$,

Then F is continuous and differentiable

Furthermore $\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

3. Describe Simpson's Rule for approximating areas. (I don't want a formula here, but rather an explanation of where the formula comes from.)



Instead of approximating areas with rectangles or trapezoids we use parabolas.

4. Find the following:

$$\frac{d}{dx}\int_x^3 \frac{3^u\pi - e}{\sqrt{u^3+7}}\,du = \frac{d}{dx}\left[-\int_3^x \frac{3^u\pi - e}{\sqrt{u^3+7}}\,du\right]$$

$$-\frac{d}{dx}\int_3^x \frac{3^u\pi - e}{\sqrt{u^3+7}}\,du = -\frac{3^x\pi - e}{\sqrt{x^3+7}}$$

by FTC1

$$\frac{d}{dx}\int_0^{x^2+3x} e^{t^2}\,dt$$

Chain rule: outside $\int_0^u e^{t^2}\,dt$

inside x^2+3x

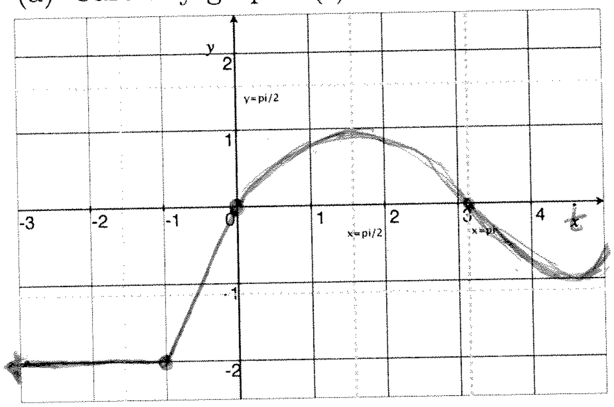
outside'(inside) · inside'

$$e^{(\text{inside})^2}\cdot(2x+3) = e^{(x^2+3x)^2}\cdot(2x+3)$$

5. Let v be the function that records the velocity of a particle which is well approximated by the following formula.

$$v(t) = \begin{cases} -2 & t \le -1 \\ 2t & \text{if } -1 \le x \le 0 \\ \sin t & \text{if } 0 < t \end{cases}$$

(a) Carefully graph $v(t)$ on the set of axis.

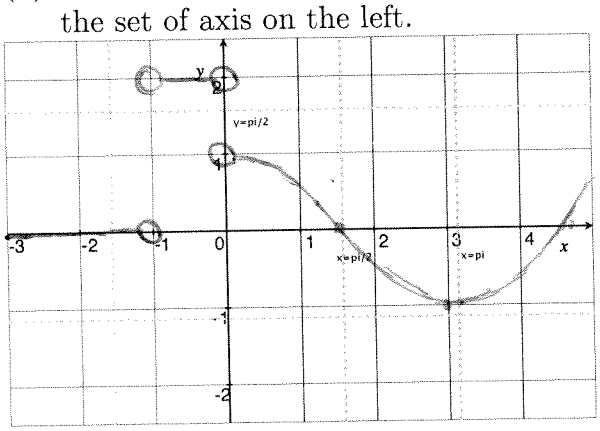


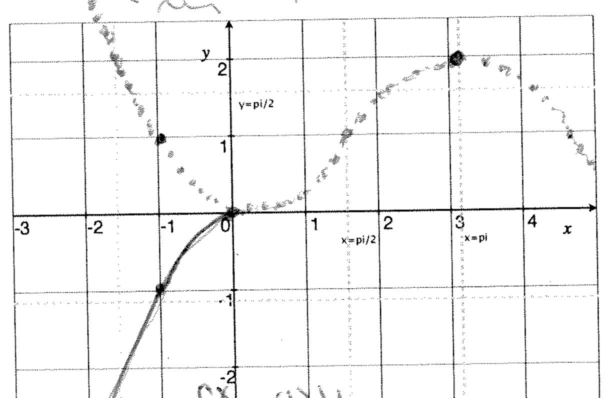
(b) Give a rough sketch of the function recording the acceleration of the particle on the set of axis on the left.

like parabola

cos function

acceleration $= v'(t)$





if $\int_{-3}^{x} v(t)dt$

(c) Give a rough sketch of the graph $\int_0^x v(t)\,dt$ on the set of axis on the right. dotted curve

(d) Describe the physical meaning of $\int_0^x v(t)\,dt$.

The distance traveled from the initial position.

consider when $x = -1$

$$\int_0^{-1} v(t)\,dt = -\int_{-1}^{0} v(t)\,dt$$

when $x = \pi$

$$\int_0^{\pi} v(t)dt = -\cos t\Big]_0^{\pi} = -[-1-1] = 2$$

Detailed answers attached ~~at~~ end

6. For each of the following outline the method(s) you would use to find the general antiderivative. *For extra credit*, find the general antiderivative (each one will earn 1%).

$$\int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx = \int_0^{\pi/4} \sec^2 x \tan^4 x \sec^2 x \, dx$$

reserve $\sec^2 x dx$ for du and let $\tan x = u$.

Use pythagorus $\tan^2\theta + 1 = \sec^2\theta$ to turn the remaining $\sec^2 x$ factor into something involving $\tan^2 x$.

Change the limits to $\int_0^{\tan \pi/4} (u^2+1) u^4 du$ & finish with FTC2.

$$\int x \cos^2 x \, dx$$

integration by parts coupled with trig identity

$= \int x \frac{1}{2}(1+\cos 2x) dx$

$= \underbrace{\int \frac{1}{2} x dx}_{\text{easy to finish}} + \int \frac{1}{2} x \cos 2x dx$

$u = x \quad v = \frac{1}{2}\sin 2x$

$du = dx \quad dv = \cos 2x dx$

which gets to a place that's easy to integrate

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{t \to \infty} \underbrace{\int_1^t \frac{1}{x^2} dx}_{\text{easy to integrate with power rule}}$$

$= \lim_{t\to\infty} \left.\frac{-1}{x}\right]_1^t$ use FTC2 ~~tcb~~

then evaluate the limit

$$\int \frac{1}{x^2\sqrt{x^2+4}} dx$$

trigonometric substitution

$x = 2\tan\theta$

$dx = 2\sec^2\theta d\theta$

Use pythagorus & restrict θ so that $\sqrt{\tan^2\theta + 1} = \sec\theta$

We'll have something like $\int \frac{\sec^2\theta}{\tan^2\theta \sec\theta} d\theta$

We can simplify to just $\sin\theta$ & $\cos\theta$'s to finish integration with subing.

Then make a ~~A~~ appropriate Δ to put answer in terms of x

$$\int_0^3 \frac{1}{x-1} dx$$

note $\frac{1}{x-1}$ is not con't at 1

$= \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$

$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1} dx + \lim_{s \to 1^+} \int_s^3 \frac{1}{x-1} dx$

use substitution

$w = x-1 \qquad u = x-1$

and ln's to integrate.

Finally evaluate the limits

$$\int \frac{17x-1}{2x^2+3x-2} dx$$

partial fractions

find A and B so that

$\frac{A}{2x-1} + \frac{B}{x+2} = \frac{17x-1}{(2x-1)(x+2)}$

Then use substitution and ln's to finish

$\int \frac{A}{2x-1} dx + \int \frac{B}{x+2} dx$

$w = 2x-1 \qquad u = x+2$

7. Let $g(x) = \dfrac{12x}{x^2+x-2}$. Find the average value of g on the interval $[2,5]$.

$(\text{ave value})(5-2) = \int_2^5 g(x)\,dx$

$\Rightarrow \text{ave value} = \frac{1}{3}\int_2^5 \frac{12x}{x^2+x-2}\,dx$

$= \frac{1}{3}\int_2^5 \frac{12x}{(x+2)(x-1)}\,dx$

$= \frac{1}{3}\int_2^5 \frac{8}{x+2} + \frac{4}{x-1}\,dx$

$w = x+2 \Rightarrow dw = dx$ $\qquad u = x-1$, $du = dx$

$\frac{1}{3}\left[\int \frac{8}{w}\,dw + \int \frac{4}{u}\,du\right]$

$\frac{1}{3}\left[8\ln|w| + 4\ln|u|\right] + c$

$\frac{1}{3}\left[8\ln|x+2| + 4\ln|x-1|\right]_2^5$

$\frac{1}{3}[8\ln 7 + 4\ln 4] - \frac{1}{3}[8\ln 4 + 4\ln 1]$

Partial fractions

$\frac{A}{x+2} + \frac{B}{x-1} = \frac{12x}{(x+2)(x-1)}$

$\frac{A(x-1) + B(x+2)}{(x+2)(x-1)} = \frac{12x}{(x+2)(x-1)}$

$\Rightarrow Ax - A + Bx + 2B = 12x$

$\Rightarrow \begin{cases} Ax + Bx = 12x \\ -A + 2B = 0 \end{cases}$ $A = 2B$

$2Bx + Bx = 12x$

$3Bx = 12x$

$\Rightarrow B = 4 \Rightarrow A = 8$

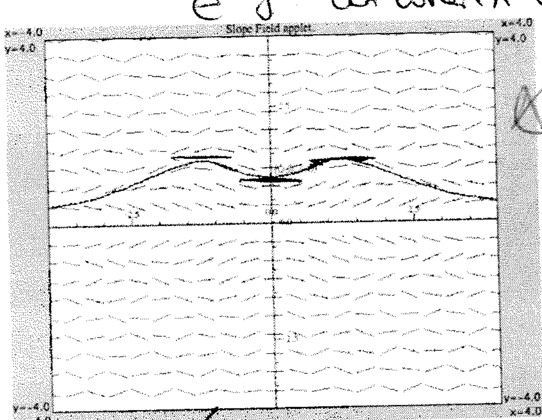
8. Match the differential equations with the solutions graphs. Briefly *justify* your choice.

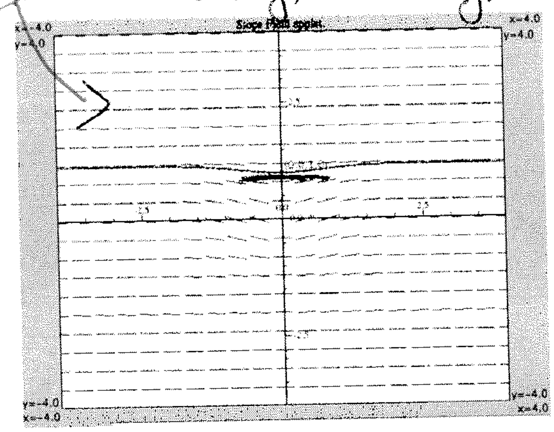
(a) $y' = xe^{-x^2-y^2} = xe^{-(x^2+y^2)} = \dfrac{x}{e^{x^2+y^2}}$

note y' only zeros at when $x=0$

(b) $y' = \sin(xy)\cos(xy)$

note y' zeros at when ever $\sin(xy)$ or $\cos(xy)$ is zero which happens more than once





several values have $y'=0$

9. Write the following in sigma notation:

$-\frac{1}{3} + \frac{3}{7} - \frac{1}{2} + \frac{5}{9} - \frac{3}{5} + \frac{7}{11}$

$= -\frac{2}{6} + \frac{3}{7} - \frac{4}{8} + \frac{5}{9} - \frac{6}{10} + \frac{7}{11}$

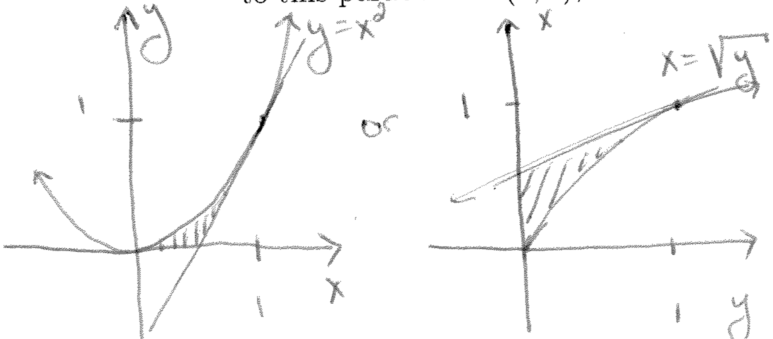
$\sum_{i=1}^{6} (-1)^i \frac{(i+1)}{(i+5)}$

note: this is one of many answers.

$1+2+4+8+16+32 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$\sum_{i=0}^{5} 2^i$

10. Let $f(x) = \sin(x)$. Find the area of the region bounded by f, $y = x^2$, the tangent line to this parabola at $(1,1)$, and the x-axis.



I'll integrate with respect to y

$$\int_0^1 \text{tangent line} - \sqrt{y}\, dy$$

$$\text{by } (*) = \int_0^1 \left(\tfrac{1}{2}y + \tfrac{1}{2}\right) - \sqrt{y}\, dy$$

$$= \tfrac{1}{4}y^2 + \tfrac{1}{2}y - \tfrac{2}{3}y^{3/2}\Big]_0^1 = \tfrac{1}{4} + \tfrac{1}{2} - \tfrac{2}{3}$$

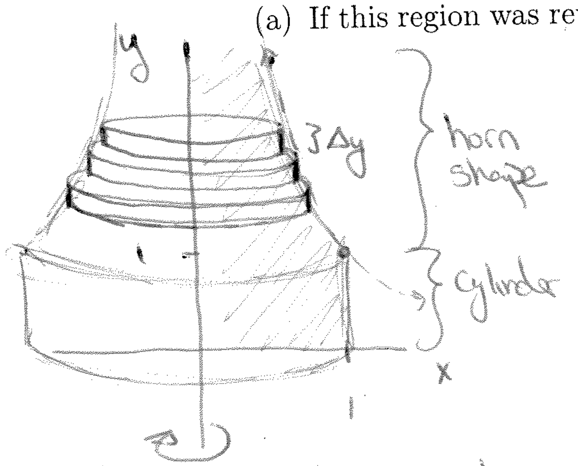
eq of tangent line $y = mx + b$

$m = f'(1) = 2x\big|_{x=1} = 2$ then $(1,1) \Rightarrow 1 = 2(1) + b \Rightarrow b = -1$

$(*)$: $y = 2x - 1$, $y + 1 = 2x$, $\tfrac{1}{2}y + \tfrac{1}{2} = x$

11. Consider the region trapped between $f(x) = \frac{1}{x}$, the x-axis, and from $x = 0$ to $x = 1$.

(a) If this region was revolved about the y-axis, what would the resulting volume be?



Vol of cylinder + Vol horn shape

$\pi 1^2 \cdot 1$ + sum of small cylinders (taken to limit)

π + sum of $\pi(\text{radius})^2 \Delta y$

π + sum of $\pi(\text{x coord})^2 \Delta y$

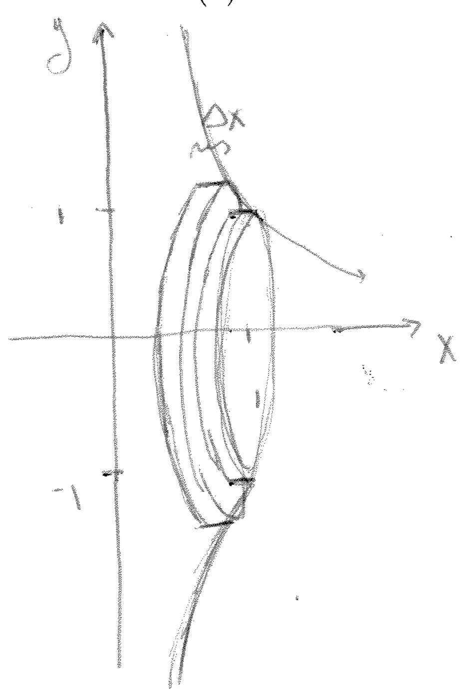
$$\pi + \int_1^\infty \pi\left(\frac{1}{y}\right)^2 dy = \pi + \pi\int_1^\infty \frac{1}{y^2}\, dy$$

$$\pi + \pi \lim_{t\to\infty} \int_1^t \frac{1}{y^2}\, dy = \pi + \pi \lim_{t\to\infty} \frac{-1}{y}\Big]_1^t = \pi + \pi\left[\lim_{t\to\infty}\frac{-1}{t} - \frac{-1}{1}\right]$$

$$= \pi + \pi \cdot 1 = 2\pi$$

$y = \frac{1}{x} \Rightarrow xy = 1 \Leftrightarrow x = \frac{1}{y}$

(b) What would its volume be if it was revolved about the x-axis?



Sum of small cylinders (taken to limit)

Sum of $\pi(\text{radius})^2 \Delta x$

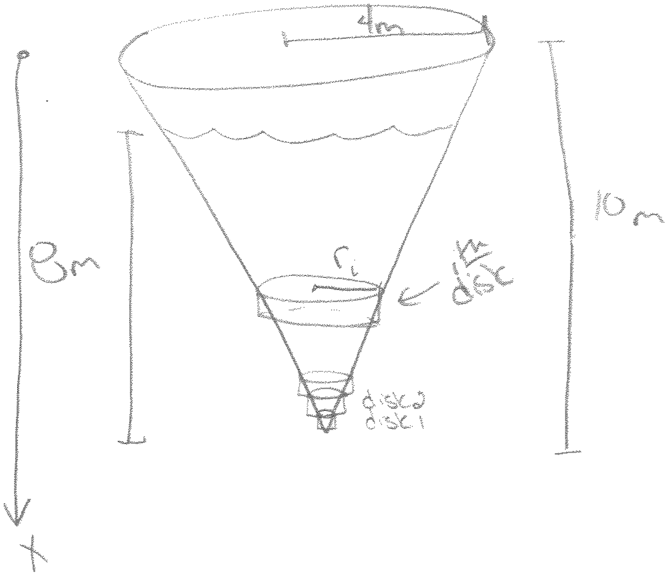
Sum of $\pi(\text{y coord})^2 \Delta x$

$$\int_0^1 \pi\left(\frac{1}{x}\right)^2 dx = \lim_{t\to 0^+} \int_t^1 \pi \frac{1}{x^2}\, dx$$

$$= \pi \lim_{t\to 0^+} \frac{-1}{x}\Big]_t^1 = \pi\left[\frac{-1}{1} - \lim_{t\to 0^+}\frac{-1}{t}\right] \quad -\infty$$

Diverges.

11. [10] A tank has the shape of an inverted circular cone with height 10m and base 4 m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000kg/m^3.)



Work $\approx$ work to move disk 1 up + work to move disk 2 up + ... + work to move last disk up

Work $= \lim_{\#\text{ of disks} \to \infty} \Big[$ work to move disk 1 up + ... + work to move last disk up $\Big]$

Work to move the i^{th} disk up = force to move i^{th} disk $\cdot$ dist

$= \left(\text{mass of } i^{th} \text{ disk}\right)\left(\text{acceleration}\right) \cdot x_i$

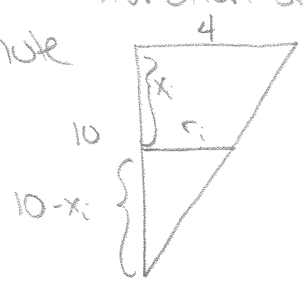
$= \left(\text{Volume of } i^{th} \text{ disk} \cdot 1000\right)(9.8) \cdot x_i$

$= \pi r_i^2 \cdot \Delta x \; 9800 \cdot x_i$

$= \pi \left(4 - \frac{2}{5} x_i\right)^2 \Delta x \cdot 9800 x_i$

need to write r_i as a function of x

note



by similar triangles $\frac{4}{r_i} = \frac{10}{10-x_i}$

$\Rightarrow 40 - 4x_i = r_i \cdot 10$

$\Rightarrow r_i = 4 - \frac{2}{5} x_i$

Returning to total work:

$$\int_2^{10} \pi\left(4 - \frac{2}{5}x\right)^2 9800x \cdot dx = 9800\pi \int_2^8 x\left(4 - \frac{2}{5}x\right)^2 dx$$

$$= 9800\pi \int_2^8 16x - \frac{16}{5}x^2 + \frac{4}{25}x^5 \, dx = 9800\pi \left[8x^2 - \frac{16}{15}x^3 + \frac{4}{5^2 \cdot 6}x^6\right]_2^8 \ldots$$

13. Dr. Card is found dead in his office at 5:00pm one evening. The temperature of his body was 80.0°F. One hour later, at 6:00pm, the body has cooled to 75.0°F. The room is kept at a constant temperature of 70°F. Assume Dr. Card had a normal temperature of 98.6°F at the time of death.

Let $f(t)$ be the temperature of the body after t hours.

(a) By Newton's law of cooling, the rate a body cools is proportional to the difference in temperature between the body and the ambient temperature. Write down the differential equation reflecting this particular situation.

$$df/dt = K(f(t)-70) \quad \text{or} \quad dy/dt = K(y-70)$$

(b) Solve for $f(t)$ as a function of t. .

$$\frac{dy}{dt} = K(y-70)$$

$$\frac{1}{y-70}\,dy = Kdt$$

$$\int \frac{1}{y-70}\,dy = \int Kdt$$

$u = y-70$

$du = dy$

$$\int \frac{1}{u}\,du = \int Kdt$$

$$\ln|u| = Kt + c$$

$$\ln|y-70| = Kt + c$$

$$|y-70| = e^{Kt+c}$$

Since the temp of the body will always be greater than 70...

$$y - 70 = e^{Kt}e^{c}$$

$$y = e^{c}e^{Kt} + 70$$

let $e^{c} = A$

$$y = Ae^{Kt} + 70$$

~~Recall~~ Let t be time since 5pm

So when $t=0$ $f(0) = 80°$

and when $t=1$ $f(1) = 75°$

$f(0) = 80°$

$$\Rightarrow 80 = Ae^{K(0)} + 70 = Ae^{0} + 70$$

$$\Rightarrow 80 = A + 70$$

$$10 = A$$

So $y = 10e^{Kt} + 70$

$f(1) = 75°$

$$\Rightarrow 75 = 10e^{K\cdot 1} + 70$$

$$5 = 10e^{K} \Rightarrow \frac{1}{2} = e^{K}$$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = K$$

So $y = 10e^{t\ln(1/2)} + 70$

$$y = 10\left(\tfrac{1}{2}\right)^{t} + 70$$

(c) When did the murder take place?

we need to find t so that

$$98.6 = 10\left(\tfrac{1}{2}\right)^{t} + \cancel{70}$$

$$-70 \qquad \cancel{-70}$$

$$\frac{28.6}{10} = \frac{\cancel{10}\left(\frac{1}{2}\right)^{t}}{\cancel{10}}$$

$$2.86 = \frac{1}{2}^{t}$$

$$\ln 2.86 = t\ln\tfrac{1}{2}$$

$$t = \frac{\ln 2.86}{\ln\frac{1}{2}} \approx -1.5$$

so around 3:30pm

#6 written up

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

8. [10 *each*] Evaluate the following if they exist.

(a) $\int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx$ $= \int_0^{\pi/4} \tan^4 x (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \int_0^1 u^4(u^2+1)\, du = \int_0^1 u^6 + u^4 \, du$

$= \frac{1}{7}u^7 + \frac{1}{5}u^5 \Big]_0^1$

$= \frac{1}{7} + \frac{1}{5} = \frac{5+7}{35} = \frac{12}{35}$

(b) $\int x \cos^2 x \, dx$ $= \int x \frac{1}{2}[1 + \cos 2x] dx = \frac{1}{2}\int x + x\cos 2x \, dx$

$= \frac{1}{2}\left[\int x\,dx + \int x\cos 2x\,dx\right] = \frac{1}{2}\left[\frac{1}{2}x^2 + x\frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right.$

$u = x$ $\quad v = \frac{1}{2}\sin 2x$

$du = dx$ $\quad dv = \cos 2x \, dx$

$= \frac{1}{4}x^2 + x\frac{1}{4}\sin 2x - \frac{1}{4}\int \sin 2x\,dx$

$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{2}\cdot\frac{1}{2}\cos 2x +$

(c) $\int_1^\infty \frac{1}{x^2}\,dx$ $= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \frac{-1}{x}\Big]_1^t$

$= \lim_{t\to\infty}\left(\frac{-1}{t} + \frac{+1}{1}\right) = \lim_{t\to\infty}\left[\frac{-1}{t} + 1\right]$

$= \lim_{t\to\infty} \frac{-1}{t} + \lim_{t\to\infty} 1$

$= 0 + 1 = 1$

#6 written up con't.

$\sin^2\theta + \cos^2\theta = 1$

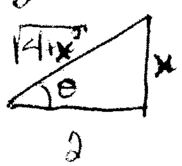
$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Sohcahtoa

(d) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \cdot 2\sec^2\theta\,d\theta$

$\frac{x}{2} = \tan\theta$



$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta\,\cancel{2}\cancel{\sec\theta}}\,d\theta \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \quad u = \sin\theta,\ du = \cos\theta\,d\theta \Rightarrow \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\cdot\frac{-1}{u} + c = \frac{-1}{4\sin\theta} + c$$

$$= \frac{-1}{4}\csc\theta + c = \frac{-1}{4}\frac{\sqrt{4+x^2}}{x} + c$$

oops

(e) $\cancel{\int_1^\infty \frac{1}{x^2}\,dx}$ $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ instead

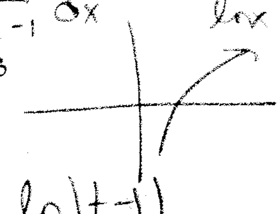
note $\frac{1}{x-1}$ is not cont at $x=1$

$$\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx = \lim_{t\to1^-}\int_0^t \frac{1}{x-1}\,dx + \lim_{s\to1^+}\int_s^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t\to1^-} \ln|x-1|\Big]_0^t + \lim_{s\to1^+} \ln|x-1|\Big]_s^3$$

$$= \underbrace{\lim_{t\to1^-}\left[\ln|t-1| - \cancel{\ln 1}\right]}_{\text{diverges}} + \lim_{s\to1^+}\left[\ln 2 - \ln|s-1|\right]$$

so $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ $\boxed{\text{diverges}}$



note

$$\lim_{t\to1^-} \ln|t-1| = \ln\left(\lim_{t\to1^-}|t-1|\right) \to -\infty$$

(f) $\displaystyle\int \frac{17x-1}{2x^2+3x-2}\,dx$

$(2x-1)(x+2)$

$$\frac{A}{2x-1} + \frac{B}{x+2} = \frac{17x-1}{(2x-1)(x+2)}$$

$u = 2x^2+3x-2$

$du = 4x+3\,dx$

$\Rightarrow Ax + 2A + B2x - B = 17x - 1$

$\Rightarrow A + 2B = 17$, $2A - B = -1$

$B = 2A+1$

$A + 2(2A+1) = 17 \Rightarrow 5A + 2 = 17$

$\Rightarrow 5A = 15$

$A = 3$

$$\int \frac{17x-1}{(2x-1)(x+2)}\,dx = \int \frac{3}{2x-1} + \frac{7}{x+2}\,dx = 3\int \frac{1}{2x-1}\,dx + 7\int \frac{1}{x+2}\,dx$$

$u = 2x-1$

$du = 2dx$

$$= 3\int \frac{1}{u}\cdot\frac{1}{2}\,du + 7\ln|x+2| + c$$

$$\boxed{= \frac{3}{2}\ln|2x-1| + 7\ln|x+2| + c}$$