EXAM 2 **TMath 125** PRACTICE

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

   T (F) If $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f" is continuous, we cannot evaluate $\int_1^4 x f"(x)\,dx$.

   $u = x$, $v = f'(x)$, $du = dx$, $dv = f''(x)$; $= [xf'(x)]_1^4 - \int_1^4 f'(x)\,dx = [4f'(4) - 1f'(1)] - [f(x)]_1^4 = [4\cdot 3 - 5] - [f(4) - f(1)] = 7 - [7-2] = 7 - 5 = 2$; work to the right → (IBP)

   T (F) Substitution yeilds: $\int_0^1 y(y^2+1)^5 dy = \int_0^1 \frac{1}{2} u^5 du$

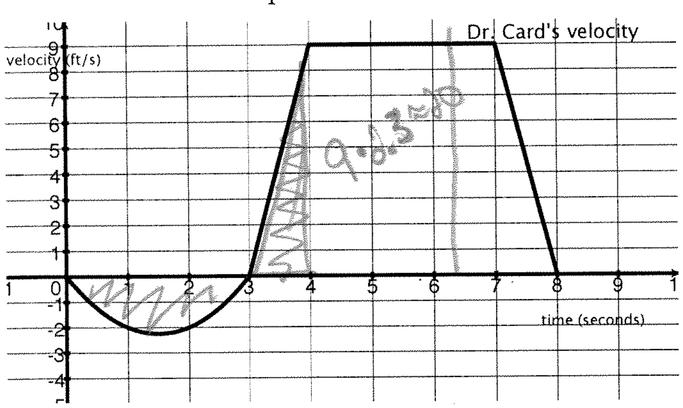
   $u = y^2+1$, $du = 2y\,dy$; The endpoints need to change.

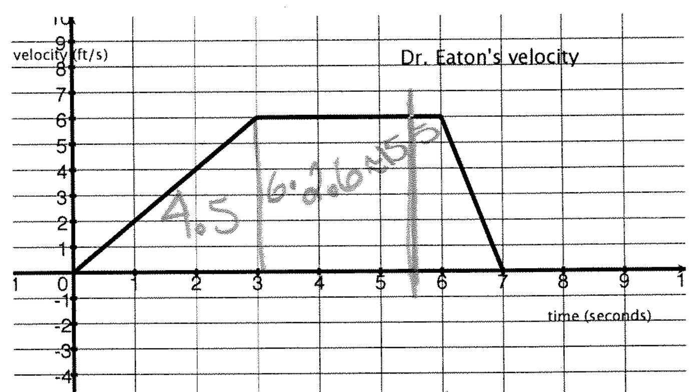
   T (F) $\int_{-1}^{1} \frac{1}{x^2} dx = \frac{-1}{x}\Big|_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$

   area should be positive. The FTC doesn't work on non cont functions

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. () Dr. Card and Dr. Eaton decide to have a short race. The following is a graph of their respective *velocities* at time t measured in seconds.





(a) [2] Estimate the ~~net~~ total distance each one runs during the race.

Recall $\frac{d}{dt}$(position) = vel ⇒ position = $\int$ vel dt; Dr. Card ≈ 4 ft (in the wrong direction) $+ \frac{9}{2} + 3\cdot 9 + \frac{9}{2} = 40$ ft; Dr. Eaton $= 3\cdot 3\cdot\frac{1}{2} + 3\cdot 6 + 1\cdot 6\cdot\frac{1}{2} = 25.5$

(b) [2] If the race is 20 ft, who wins the race? Explain how you know.

We want to know who travels 20 ft in the positive direction first. ie what t is ∋ $\int_0^t \text{vel}\,dt = 20$. By inspection it takes Dr. Eaton ≈ 5.6 sec ← winner. Dr Card starts backwards & has to make up for lost ground in the 3rd–4th sec

$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$ $\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$ $\sin(x+y) = \cos x \sin y + \cos y \sin x$

$\sin(2x) = 2\cos x \sin x$

$= \frac{\cos x \cos x}{\sin^2 x} + \frac{-\sin x}{\sin x}$ $1 + \cot^2 x = \csc^2 x$

$= \frac{-\cos^2 x - \sin^2 x}{\sin^2 x}$

$= \frac{-1}{\sin^2 x} = -\csc^2 x$

3. For each of the following outline the method(s) you would use to find the general antiderivative. *For extra credit*, find the general antiderivative (each one will earn 1%).

$$\int x4^x\,dx$$

integration by parts

$u = x$

$dv = 4^x$

$$\int e^{\cos(t)} \sin(2t)\,dt$$

double angle identity $\Rightarrow \int e^{\cos(t)} 2\cos(t)\sin(t)\,dt$

substitution $w = \cos t$ $dw = -\sin(t)\,dt$

integrate by parts $\Rightarrow \int e^w 2w(-1)\,dw$

$u = w$

$dv = e^w$

typo $\rightarrow$

$$\int 3\cot^3 x\,dx$$

use Pythagorus to introduce $\csc x$

$\Rightarrow \int 3\cot x(\csc^2 x - 1)\,dx$

split into 2 integrals (w/ algebra)

$\int 3\cot x \csc^2 x\,dx - \int 3\cot x\,dx$

$\int 3\frac{\cos x}{\sin x}\,dx$

let $u = \cot x$ $\Rightarrow du = \csc^2 x\,dx$ substitution will finish

let $w = \sin x$ $du = \cos x\,dx$ sub will finish it

$$\int \pi \left(\frac{36}{x^2+36}\right)^2 dx$$

pull out the constant π and 36^2

$\Rightarrow \pi 36^2 \int \frac{1}{x^2+36}\,dx$

factor out a $1/36$

$\Rightarrow \pi 36^2 \int \frac{1}{36} \cdot \frac{1}{\frac{x^2}{36}+1}\,dx$

$\Rightarrow 36\pi \int \frac{1}{(\frac{x}{6})^2+1}\,dx$ let $u = \frac{x}{6}$ and use arctan

typo $\rightarrow$

$$\int \frac{x^4}{x-1}\,dx$$

use long division to break it up into

$\int \text{polynomial} + \frac{R}{x-1}\,dx$

where R is the remainder.

Then use substitution $u = x-1$ and ln's to integrate

$$\int \frac{x^3-4x-10}{x^2-x-6}\,dx$$

use long division to break it up into

$\int \text{polynomial} + \frac{R}{x^2-x-6}\,dx$

easy to integrate

use partial fractions to break up into

$\frac{A}{(x-3)} + \frac{B}{(x+2)}$

Then use substitution and ln's to integrate.

complete worked out problems follow on the next few pages.

Page 2 extra credit work

$\int x\, 4^x\,dx$ IP

$u = x \qquad v = \frac{1}{\ln 4} 4^x$

$du = dx \qquad dv = 4^x$

$x \frac{1}{\ln 4} 4^x - \int \frac{1}{\ln 4} 4^x dx$

$= \frac{x 4^x}{\ln 4} - \frac{1}{\ln 4} \int 4^x dx$

$= \frac{x 4^x}{\ln 4} - \frac{1}{\ln 4} \cdot \frac{1}{\ln 4} 4^x + c$

ck: $\left[\frac{x}{\ln 4} \cdot 4^x - \left(\frac{1}{\ln 4}\right)^2 4^x + c\right]'$ (product rule)

$= \frac{x}{\ln 4} \cdot 4^x \ln 4 + \frac{1}{\ln 4} \cdot 4^x - \left(\frac{1}{\ln 4}\right)^2 4^x \ln 4$

$= x 4^x + \frac{4^x}{\ln 4} - \frac{4^x}{\ln 4}$ ✓

$\int e^{\cos(2t)} \sin(2t)\,dt$

since $\sin 2t = 2\cos t \sin t \Rightarrow \int e^{\cos(t)} 2\cos t \sin t\,dt$

let $w = \cos t$ & $dw = -\sin t\,dt$ $\Rightarrow -dw = \sin t\,dt$ $\Rightarrow \int e^w 2w(-1)\,dw$

simplify $-2\int w e^w dw$

integrate by parts $= -2\left[we^w - \int e^w dw\right]$

$u = w \qquad v = e^w$

$du = dw \qquad dv = e^w dw$ $\quad = -2we^w + 2e^w + c$

$= -2\cos t\, e^{\cos t} + 2e^{\cos t} + c$

ck $\left[-2\cos t\, e^{\cos t} + 2e^{\cos t} + c\right]'$ (product rule)

$= -2\cos t\, e^{\cos t} \cdot -\sin t + 2\sin t\, e^{\cos t} + 2e^{\cos t}(-\sin t)$

$= 2\cos t \sin t\, e^{\cos t} = \sin(2t) e^{\cos t}$ ✓ by double angle id

$\int 3\cot^3 x\,dx = \int 3\cot x(\csc^2 x - 1)\,dx$ (by pythagorus)

$= \int 3\cot x \csc^2 x - 3\cot x\,dx$

$= 3\int \cot x \csc^2 x\,dx - 3\int \cot x\,dx$

$= 3\int \cot x \csc^2 x\,dx - 3\int \frac{\cos x}{\sin x}dx$

$u = \cot x \qquad w = \sin x$

$du = -\csc^2 x\,dx \qquad dw = \cos x\,dx$

$\Rightarrow -du = \csc^2 x\,dx$

$= 3\int u(-1)\,du - 3\int \frac{1}{w}\,dw$

$= -3\frac{1}{2}u^2 - 3\ln|w| + c$

$= -\frac{3}{2}\cot^2 x - 3\ln|\sin x| + c$

ck: $\left[-\frac{3}{2}\cot^2 x - 3\ln|\sin x| + c\right]'$

$= -\frac{3}{2} \cdot 2\cot x \cdot (-\csc^2 x) - 3\frac{1}{\sin x} \cdot \cos x$

$= 3\cot x \csc^2 x - 3\cot x$

$= 3\cot x(\csc^2 x - 1)$ ✓

page 2 extra credit work

$$\int \pi\left(\frac{36}{x^2+36}\right)^2 dx = \pi\int 36^2 \frac{1}{x^2+36}dx$$

$$= \pi 36^2 \int \frac{1}{36\left[\frac{x^2}{36}+1\right]}dx = \frac{\pi 36^2}{36}\int \frac{1}{\left(\frac{x}{6}\right)^2+1}dx$$

$$= \pi 36 \int \frac{1}{\left(\frac{x}{6}\right)^2+1}dx \qquad u = \frac{x}{6} \qquad du = \frac{1}{6}dx$$

$$= 36\pi \int \frac{1}{u^2+1} 6du$$

$$= 36\cdot 6\cdot \pi \arctan u + c$$

$$= 216\pi \arctan\left(\frac{x}{6}\right) + c$$

Ck: $\left[216\pi \arctan(x/6) + c\right]'$

$$216\pi \frac{1}{\left(\frac{x}{6}\right)^2+1}\cdot\frac{1}{6} + 0$$

$$= 36\pi \frac{1}{\left(\frac{x}{6}\right)^2+1} = \frac{36\pi}{\frac{x^2}{36}+1}\frac{36}{36}$$

$$= \frac{36^2\pi}{x^2+36} \checkmark$$

$36 \cdot 6 = 216$

note: I'm fairly certain trigonometric substitution would have worked on this problem as well.

$$\int \frac{x^4}{x-1}dx$$

$$x-1 \overline{)\, x^4+0x^3+0x^2+0x+} \quad \text{quotient } x^3+x^2+x+1$$
$$-(x^4-x^3)$$
$$x^3+0x^2$$
$$-(x^3-x^2)$$
$$x^2+0x$$
$$-(x^2-x)$$
$$x+0$$
$$-(x-1)$$
$$R\,1$$

$$= \int x^3+x^2+x+1+\frac{1}{x-1}dx$$

$$= \frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x+x+\int\frac{1}{x-1}dx$$

$u = x-1$

$du = dx$

$$= \frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+c$$

Ck: $\left[\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+c\right]'$

$$= x^3+x^2+x+1+\frac{1}{x-1}+0$$

$$= \frac{x^3(x-1)}{x-1}+\frac{x^2(x-1)}{x-1}+\frac{x(x-1)}{x-1}+\frac{1(x-1)}{x-1}+\frac{1}{x-1}$$

$$= \frac{x^4-x^3+x^3-x^2+x^2-x+x-1+1}{x-1}$$

$$= \frac{x^4}{x-1}$$

$$\int \frac{x^3-4x-10}{x^2-x-6}dx = \int x+1+\frac{3x-4}{x^2-x-6}dx = \int x+1+\frac{3x-4}{(x-3)(x+2)}dx$$

$$\to \int x+1+\frac{1}{x-3}+\frac{2}{x+2}dx$$

$$= \frac{1}{2}x^2+x+\ln|x-3|+2\ln|x+2|+c$$

$$x^2-x-6 \overline{)\, x^3+0x^2-4x-10} \quad \text{quotient } x+1$$
$$-(x^3-x^2-6x)$$
$$x^2+2x-10$$
$$-(x^2-x-6)$$
$$3x-4$$

$$\frac{A}{x-3}+\frac{B}{x+2} = \frac{3x-4}{(x-3)(x+2)}$$

$$\frac{A(x+2)+B(x-3)}{(x-3)(x+2)} = \frac{3x-4}{(x-3)(x+2)}$$

$$\Rightarrow Ax+2A+Bx-3B = 3x-4$$

$$\to \begin{cases} Ax+Bx=3x \\ 2A-3B=-4 \end{cases} \Rightarrow \begin{cases} A+B=3 \\ 2A-3B=-4 \end{cases}$$

$A = 3-B$ sub into 2^{nd}

$\Rightarrow 2(3-B)-3B=-4$

$\Rightarrow -5B=-10$

$B=2$

$S_o\ A=1$

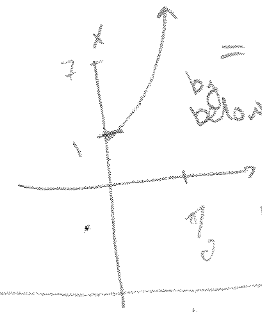
$\cos^2 u = \ldots$

$\sin^2 u = \ldots$

4. [] (§7.3) Let $f(t) = \dfrac{\sqrt{x^2 - 1}}{x}$. Find the average value of f on the interval $[1, 7]$.

$(7-1) f(\text{ave}) = \int_1^7 f(t)\,dt$

$\Rightarrow f(\text{ave}) = \frac{1}{6}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx$



$= \frac{1}{6}\left[\sqrt{x^2-1} - \arccos\left(\frac{1}{x}\right)\right]_1^7$ by below

$= \frac{1}{6}\left[\sqrt{48} - \arccos\left(\frac{1}{7}\right) + \arccos(1)\right]$ ($\arccos(1)$ = 0)

$= \frac{1}{6}\left[\sqrt{48} - \arccos\left(\frac{1}{7}\right)\right]$

~~know~~ ~~sub~~ ~~sub $u = x^2-1$~~ extra x is in den

IP maybe ..

try $x = \sec u$, $dx = \sec u \tan u\,du$ } limitation of substitution dont effect

$x = \frac{1}{\cos u} \rightarrow \cos u = \frac{1}{x}$ Sohcahtoa

$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2 u - 1}}{\sec u} \sec u \tan u\,du$

try $\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

$= \int \tan u \tan u\,du$

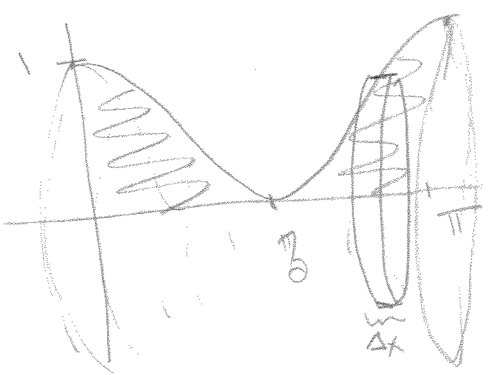
$= \int \tan^2 u\,du = \int \sec^2 u - 1\,du$

$\rightarrow \int \sec^2 u\,du - \int du$

$= \tan u - u + c$

$= \frac{\sqrt{x^2-1}}{1} - \arccos\left(\frac{1}{x}\right) + c$

5. [] (§) The region under the curve $y = \cos^2(x)$ from $0 \le x \le \pi$ is rotated about the x-axis, find the volume of the resulting solid.



looks like an hourglass on its side

Sum of approx cylinders

$= \text{Sum of } \pi r^2 \Delta x$

$= \text{sum of } \pi y^2 \Delta x = \text{sum of } \pi(\cos^2 x)^2 \Delta x$

$\int_0^\pi \pi(\cos^2 x)^2\,dx = \pi \int_0^\pi \left(\frac{1}{2}\cos(2x) + \frac{1}{2}\right)^2 dx$ (double angled)

$= \pi \int_0^\pi \frac{1}{4}\cos^2(2x) + \frac{1}{2}\cos 2x + \frac{1}{4}\,dx$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\cos(2x) = \cos^2 x - \sin^2 y$

$\cos^2 w = \cos(2w) + \sin^2 w$

$= \cos(2w) + 1 - \cos^2 w$

$2\cos^2 w = \cos(2w) + 1$

$\cos^2 w = \frac{1}{2}[\cos(2w) + 1]$

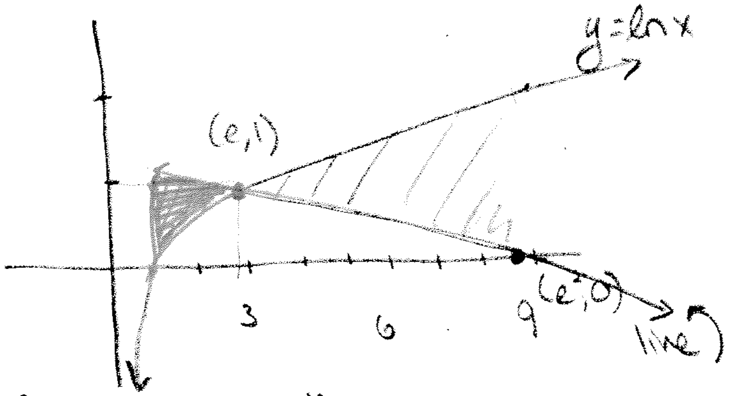
$= \pi \int_0^\pi \frac{1}{4}\left[\frac{1}{2}\cos(2(2x)) + \frac{1}{2}\right] + \frac{1}{2}\cos 2x + \frac{1}{4}\,dx$ (double angle)

$= \pi \int_0^\pi \frac{1}{8}\cos(4x) + \frac{1}{8} + \frac{1}{2}\cos 2x + \frac{1}{4}\,dx$

$u = 4x$ $\qquad w = 2x$

$= \pi\left[\frac{1}{8}\cdot\frac{1}{4}\sin 4x + \frac{1}{8}x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}x\right]_0^\pi$

6. (a) [8] Interpret $\int_1^{e^2} \ln x - \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right) dx$ as the area of a region, by sketching a graph. *Hint: $x = e$ and $x = e^2$ are good points to plot.*



Notice $\frac{-1}{e(e-1)}x + \frac{e}{e-1}$ is just $mx + b$ so we're looking at a line.

Following the hint

$\frac{-1}{e(e-1)}e + \frac{e}{e-1} = \frac{-1}{(e-1)} + \frac{e}{e-1} = \frac{-1+e}{e-1}$

and

$\frac{-1}{e(e-1)}e^2 + \frac{e}{e-1} = \frac{-e}{e-1} + \frac{e}{e-1} = 0$

The integral is the ///// area − (shaded) area shown above

(b) [4] Interpret $\int_1^{e^2} \left|\ln x - \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right)\right| dx$ as the area of a region.

The absolute values mean we'll add the two areas shown above. Thus $\int_1^{e^2} \left|\ln x - \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right)\right| dx$ is the total area trapped between $y = \ln x$ on the line $y = \frac{-1}{e(e-1)}x + \frac{e}{e-1}$.

(c) [5] Evaluate $\int_1^{e^2} \left|\ln x - \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right)\right| dx$.

$= \int_1^{e} \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right) - \ln x \, dx + \int_e^{e^2} \ln x - \left(\frac{-1}{e(e-1)}x + \frac{e}{e-1}\right) dx$

$= \left[\frac{-1}{2e(e-1)}x^2 + \frac{e}{e-1}x\right]_1^e - \int_1^e \ln x\, dx + \int_e^{e^2} \ln x\, dx + \left[\frac{1}{2e(e-1)}x^2 - \frac{e}{e-1}x\right]_e^{e^2}$

$= \left[\frac{-x^2}{2e(e-1)} + \frac{e}{e-1}x\right]_1^e - \left[x\ln x - x\right]_1^e + \left[x\ln x - x\right]_e^{e^2} + \left[\frac{x^2}{2e(e-1)} - \frac{e}{e-1}x\right]_e^{e^2}$

$= \frac{-e}{2(e-1)} + \frac{e^2}{e-1} + \frac{1}{2e(e-1)} - \frac{e}{e-1} - [e - e + 1]$

$+ [e^2 \cdot 2 - e^2 - e(1) - e] + \left[\frac{e}{2(e-1)} - \frac{e^3}{e-1} - \frac{e}{2(e-1)} + \frac{e^2}{e-1}\right]$

$\int \ln x\, dx = x\ln x - \int x \frac{1}{x} dx$

$u = \ln x \quad v = x$

$du = \frac{1}{x}dx \quad dv = dx$

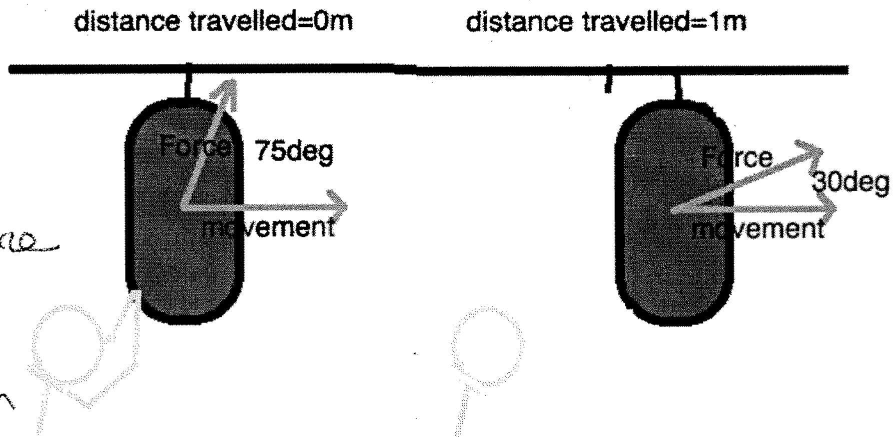
$\int \ln x\, dx = x\ln x - \int 1\, dx$

$= x\ln x - x + C$

Ck: $x\frac{1}{x} + (1)\ln x - 1$ ✓

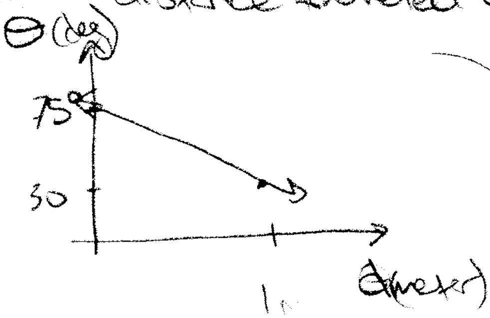
7. () A factory worker is trying to push a large package suspended from a track on the ceiling a meter to the right. Conveniently the worker's are length is 1 meter and she can apply 130 Newtons to do so. However, given her short height she can only apply the force at an angle. Initially she can only push the package up and to the right making an angle of 75° with the horizontal, but by the end of the 1 meter she has a better angle of 30° (picture attempted below). Assume the angle varies ~~directly~~ linearly with the distance that the package travels. How much work does the factory worker do on the object?

fix →



let Θ be the angle between movement + force applied

B/c Θ varies linearly with distance traveled (d)



$$\Theta = \frac{75-30}{1} d + 75$$

$$\Theta = 45d + 75$$

person force, Θ

Sohcahtoa

$$\cos\Theta = \frac{\text{Force in direction d}}{\text{Force of person}}$$

$\Rightarrow$ (Force of person) $\cos\Theta$ = Force in direction d

$$\text{work} = \int \text{Force } dd$$

The force shown above is not being applied in the same direction as the movement. Thus we are not looking at $\int_0^1 130\, dd$ but rather

$$\int_0^1 \text{Force applied in direction of movement } dd$$

$$= \int_0^1 (\text{Force of person}) \cos\Theta \, dd$$

$$= \int_0^1 130 \cdot \cos\Theta \, dd$$ b/c we were told person's force

$$= \int_0^1 130 \cos(45d+75)\, dd$$

$u = 45d + 75$

$du = 45\,dd$

$$= \int_{75}^{120} 130 (\cos u) \frac{1}{45} du = \frac{45}{130} \sin u \Big]_{75}^{120}$$