

Notes: #4 needs to be more interesting
 add distance problem
 #9 add 'radius' after base to
 make PRACTICE
 this matches in § 6.4

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let a , b , and c be constants. Assume f and g are continuous.

T F $\int_a^b cf(x)dx = c \int_a^b f(x) dx$

T F $\int f(x)g(x)dx = \int f(x) dx \int g(x)dx$

$\int x \cdot x dx$ vs $\int x dx \int x dx$
 $\frac{1}{3}x^3 + c$ vs $(\frac{1}{2}x^2)(\frac{1}{2}x^2) + d$

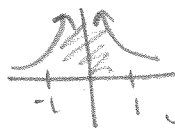
T F All continuous functions have derivatives.



T F All continuous functions have antiderivatives.

If you think of antiderivatives as areas this is believable

T F $\int_{-1}^1 \frac{1}{x^2} dx = \frac{-1}{x} \Big|_{-1}^1 = \frac{-1}{1} - \frac{-1}{-1} = -2$



area should be positive
 we can't use FTC on non cont functions

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

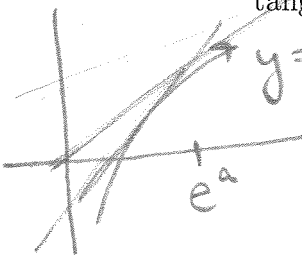
2. Oil leaked from a tank at a rate of $r(t)$ liters per hour. The rate decreased as time passed and values of the rate at two hour time intervals are shown in the table. Find lower and upper estimates for the total amount of oil that leaked out.

t (hours)	0	2	4	6	8	10
$f(t)$ (Liters/hour)	8.7	7.6	6.8	6.2	5.7	5.3

upper estimate: $2 \cdot 8.7 + 2 \cdot 7.6 + 2 \cdot 6.8 + 2 \cdot 6.2 + 2 \cdot 5.7 =$

lower estimate: $2 \cdot 7.6 + 2 \cdot 6.8 + 2 \cdot 6.2 + 2 \cdot 5.7 + 2 \cdot 5.3 =$

3. Let a be a constant (like 2.5 or something). Find the equation of the line that is tangent to the graph of $y = \ln x$ at $x = e^a$ for some constant a .



looking for $y = mx + b$ or $y - y_1 = m(x - x_1)$
 $m = \text{slope of line tangent to } y = \ln x \text{ at } x = e^a = y' \Big|_{x=e^a} = \frac{1}{x} \Big|_{x=e^a} = \frac{1}{e^a} = e^{-a}$

line passes thru $(e^a, \ln(e^a)) = (e^a, a)$ so
 $y - a = e^{-a}(x - e^a)$ or $y = e^{-a}x - 1 + a$

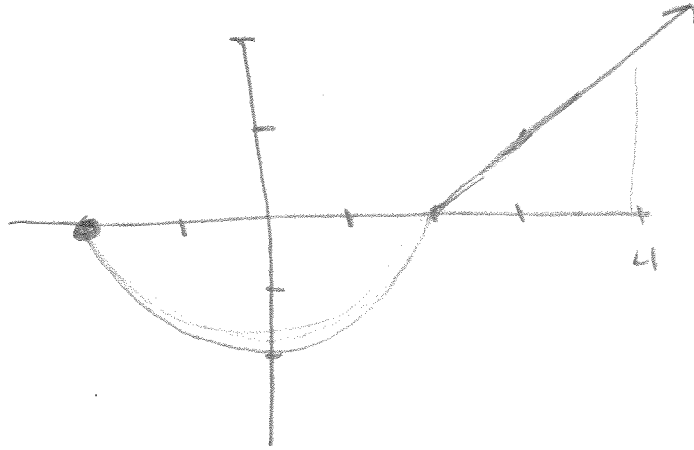
$$f(x) = \begin{cases} -\sqrt{4-x^2}; & \text{if } -2 \leq x \leq 2 \\ x-2; & \text{if } 2 < x \end{cases}$$

4. Refer to the above definition of $f(x)$ to answer the following questions.

(a) Carefully graph $f(x)$.

$y = -\sqrt{4-x^2}$
 $\Rightarrow y^2 = 4-x^2$
 $\Rightarrow x^2 + y^2 = 4$
 circle centred at $(0,0)$
 radius 2

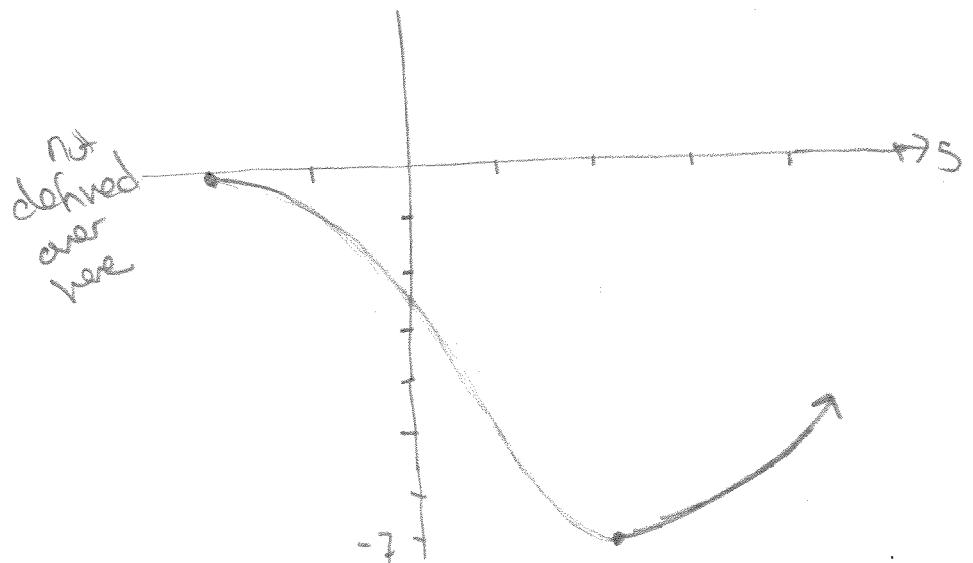
$y = x - 2$
 line with y intercept -2
 slope 1



(b) Use your above graph to find $\int_{-2}^4 f(x) dx$.

- semicircle area + triangle area
 $-\frac{1}{2} \pi (2)^2 + \frac{1}{2} \cdot 2 \cdot 2 = -2\pi + 2 = 2(1-\pi)$

(c) Give a rough sketch the graph of $\int_{-2}^x f(t) dt$.



5. Find

$$\frac{d}{dx} \int_0^{\tan x} \sqrt{1+r^3} dr$$

Chain Rule

$$g(x) = \tan x \quad g'(x) = \sec^2 x$$

$$f(u) = \int_0^u \sqrt{1+r^3} dr \quad f'(u) = \sqrt{1+u^3}$$

FTC1

$$f'(g(x))g'(x) = f'(\tan x) \cdot \sec^2 x$$

$$= \sqrt{1+(\tan x)^3} \cdot \sec^2 x$$

$$\left(\int_x^0 e^{\arctan(t)} dt \right)'$$

$$= \left(- \int_0^x e^{\arctan t} dt \right)'$$

$$= - \left(\int_0^x e^{\arctan(t)} dt \right)'$$

$$= - e^{\arctan(x)}$$

6. Evaluate the following.


$$\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1+x^6} dx = 0$$

$$\int_{-\pi/2}^{\pi/2} \sin x \left(\frac{x^2}{1+x^6} \right) dx$$

notice

$$\sin(-x) \frac{(-x)^2}{1+(-x)^6} = - \left(\sin x \frac{x^2}{1+x^6} \right)$$

There is a symmetry here ∇
(It's an odd function)



$$\int \tan(x) dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

$$u = \cos x$$

$$du = -\sin x dx$$

$$\Rightarrow -du = \sin x dx$$

$$= \int \frac{1}{\cos x} \sin x dx$$

$$= \int \frac{1}{u} (-1) du = - \int \frac{1}{u} du = - \ln|u| + C$$

$$= -\ln|\cos x| + C \text{ or } \ln|\cos x|^{-1} + C$$

$$\text{or } \ln|\sec x| + C$$

$$\int x^3 \sqrt{x^2+1} dx$$

$$u = x^2 + 1 \Rightarrow u - 1 = x^2$$

$$du = 2x dx \Rightarrow \frac{1}{2} du = x dx$$

$$\int x x^2 \sqrt{x^2+1} dx = \int x^2 \sqrt{x^2+1} x dx$$

$$= \int (u-1) \sqrt{u} \frac{1}{2} du = \frac{1}{2} \int (u-1) u^{1/2} du$$

$$= \frac{1}{2} \int u^{3/2} - u^{1/2} du = \frac{1}{2} \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right] + C$$

$$= \frac{1}{5} (x^2+1)^{5/2} - \frac{1}{3} (x^2+1)^{3/2} + C$$

$$\int_4^7 \sqrt{2t+1} dt$$

$$u = 2t + 1$$

$$du = 2 dt \Rightarrow \frac{1}{2} du = dt$$

$$\int_{2(4)+1}^{2(7)+1} \sqrt{u} \frac{1}{2} du = \frac{1}{2} \int_9^{15} u^{1/2} du$$

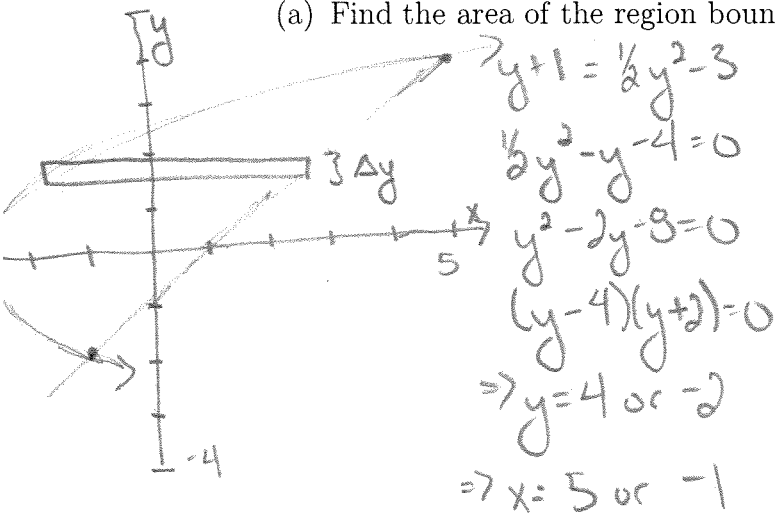
$$= \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_9^{15}$$

$$= \frac{1}{3} (15^{3/2} - 9^{3/2})$$

$$= \frac{1}{3} (15^{3/2} - 27)$$

7. Consider the line $y + 1 = x$ and the parabola $\frac{1}{2}y^2 - 3 = x$.

(a) Find the area of the region bounded by the above two curves.



$$\int_{-2}^4 (y+1) - (\frac{1}{2}y^2 - 3) dy$$

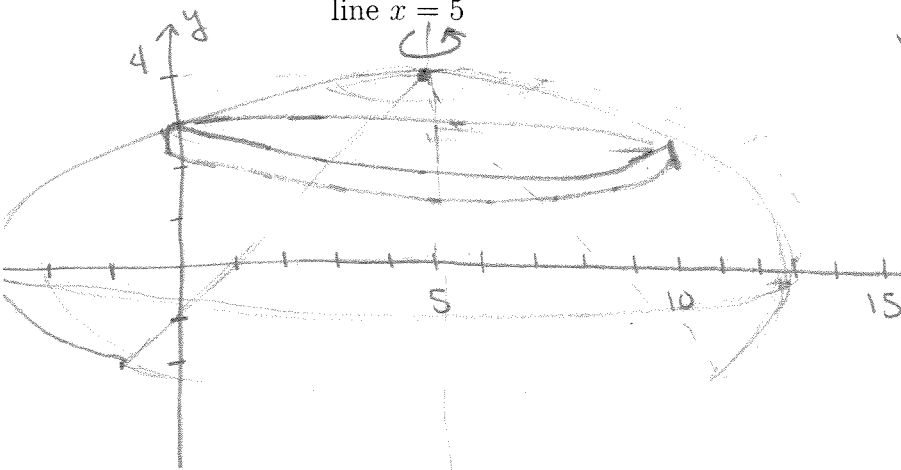
$$= \int_{-2}^4 -\frac{1}{2}y^2 + y + 4 dy$$

$$= -\frac{1}{2} \cdot \frac{1}{3} y^3 + \frac{1}{2} y^2 + 4y \Big|_{-2}^4$$

$$= \left(-\frac{1}{6} \cdot 4^3 + \frac{1}{2} \cdot 4^2 + 4 \cdot 4 \right) - \left(-\frac{1}{6} (-2)^3 + \frac{1}{2} (-2)^2 + 4(-2) \right)$$

$$= 18$$

(b) Find the volume of the solid that results by rotating the above region about the line $x = 5$



volume of - volume of

$$\pi r_o^2 \Delta y - \pi r_i^2 \Delta y$$

$$r_o = 5 - x_{\text{coord}}$$

$$= 5 - (\frac{1}{2}y^2 - 3)$$

$$= 8 - \frac{1}{2}y^2$$

$$r_i = 5 - x_{\text{coord}}$$

$$= 5 - (y+1)$$

$$= 4 - y$$

volume of - volume of

$$\pi (8 - \frac{1}{2}y^2)^2 \Delta y - \pi (4 - y)^2 \Delta y$$

$$\Rightarrow \int_{-2}^4 \pi (8 - \frac{1}{2}y^2)^2 dy - \int_{-2}^4 \pi (4 - y)^2 dy$$

$$= \int_{-2}^4 \pi \left[(8 - \frac{1}{2}y^2)^2 - (4 - y)^2 \right] dy$$

$$= \pi \int_{-2}^4 \left[64 - 8y^2 + \frac{1}{4}y^4 - 16 + 8y - y^2 \right] dy$$

$$\pi \int_{-2}^4 \left[\frac{1}{4}y^4 - 9y^2 + 8y + 48 \right] dy$$

$$\pi \left[\frac{1}{4} \cdot \frac{1}{5} y^5 - \frac{9}{3} y^3 + \frac{8}{2} y^2 + 48y \right]_{-2}^4$$

$$= \pi \left[\left(\frac{1}{20} \cdot 4^5 - 3(4)^3 + 4(4)^2 + 48(4) \right) - \left(\frac{1}{20} (-2)^5 - 3(-2)^3 + 4(-2)^2 + 48(-2) \right) \right]$$

8. Kobayashi has won the hot dog-eating world championship six times. Recently he challenged a giant bear to a 3 minute hot dog-eating contest. Kobayashi found that the rate he can eat hot dogs goes down as time goes by and can be modeled by $k(t) = \frac{12}{(t+1)^2} + 24$, where t is measured in minutes. The bear isn't quite as used to the system and seems to start with a slower rate that gets larger and is well modeled by $b(t) = 8t^3 + 20$. Find out how many hot dogs Kobayashi and the bear eat and determine who won the contest.

$$\int_0^3 k(t) dt = \text{total \# of hot dogs Kobayashi ate in 3 min}$$

$$= \int_0^3 \left(\frac{12}{(t+1)^2} + 24 \right) dt = \int_0^3 \frac{12}{(t+1)^2} dt + \int_0^3 24 dt$$

$$= 12 \int_0^3 \frac{1}{(t+1)^2} dt + 24t \Big|_0^3$$

$$u = t+1 \\ du = dt$$

$$= 12 \int_1^4 \frac{1}{u^2} du + (24 \cdot 3 - 24 \cdot 0)$$

$$= 12 \left[-\frac{1}{u} \right]_1^4 + 72 = -12 \left[\frac{1}{4} - 1 \right] + 72 = -12 \left(-\frac{3}{4} \right) + 72 = 9 + 72 = 81 \text{ hot dogs}$$

$$\int_0^3 b(t) dt = \text{total \# of hot dogs bear ate in 3 min}$$

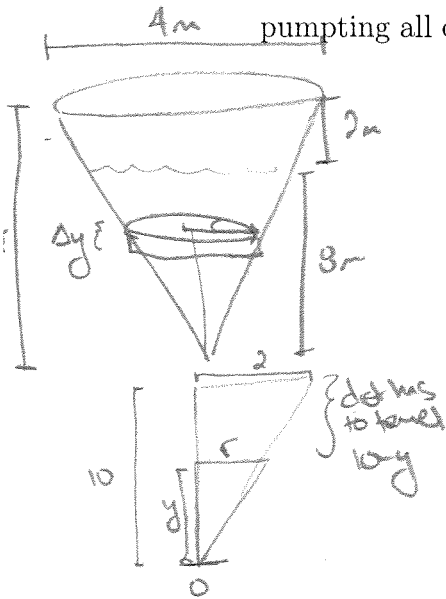
$$= \int_0^3 (8t^3 + 20) dt = \left[\frac{8}{4} t^4 + 20t \right]_0^3$$

$$= (2(3)^4 + 20(3)) - (2(0)^4 + 20(0))$$

$$= 222 \text{ hot dogs}$$

Bear wins

9. A tank has the shape of an inverted circular cone with height 10m and base 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3 .)



$$\text{Work} = \text{Force} \cdot \text{distance} = \left(\frac{\text{mass} \cdot \text{Density} \cdot \text{gravity}}{\text{kg} \cdot \frac{\text{m}}{\text{s}^2}} \right) \cdot \text{distance}$$

1) approx $\pi r^2 \Delta y$

2) relation between r & y

$$\frac{10}{2} = \frac{y}{r} \Rightarrow 5r = y$$

(similar Δ 's) $\Rightarrow r = \frac{1}{5}y$

approx disks true mass

$$\pi \left(\frac{1}{5}y \right)^2 \Delta y$$

So Total work is $\int \text{Mass} \cdot 1000 \cdot 9.8 \cdot \text{distance to travel}$

$$= \int_0^8 \pi \frac{1}{25} y^2 dy \cdot 1000 \cdot 9.8 (10-y)$$

$$\int_0^8 \pi \frac{1000 \cdot 9.8}{25} y^2 (10-y) dy$$

$$= \int_0^8 \pi 40 \cdot 9.8 (10y^2 - y^3) dy$$

$$= \pi 392 \int_0^8 (10y^2 - y^3) dy$$

$$= 392\pi \left[\frac{10}{3} y^3 - \frac{1}{4} y^4 \right]_0^8$$

$$= 392\pi \left[\frac{10 \cdot 8^3}{3} - \frac{1}{4} (8)^4 \right] = 392\pi \left[\frac{10 \cdot 512}{3} - \frac{1}{4} \cdot 4096 \right]$$

$$= 392\pi \cdot 68.3 \approx 8.4 \cdot 10^5 \text{ J}$$