Key

Exam 2 TMath 125 /40 Spring 2012

1. [4] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be continuous functions and x be a positive number.

T **(F)** $\sqrt{x^2+4} = x+2$ $\sqrt{(x+2)(x+2)} = \sqrt{(x+2)^2} = x+2$

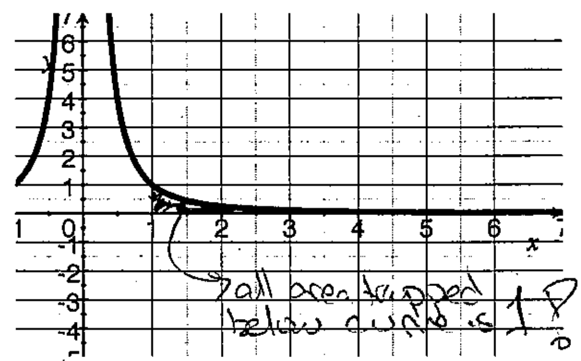
T **(F)** $\displaystyle\int \frac{\cos x}{e^x}\,dx = \frac{\sin(x)}{e^x} + c$ $\frac{d}{dx}\left(\frac{\sin x}{e^x}\right) = \frac{e^x\cos x - \sin x\, e^x}{e^{2x}} = \frac{\cos x - \sin x}{e^x} \neq \frac{\cos x}{e^x}$

(T) F The area under a force function graphed with respect to distance, is the total work.

(T) F All polynomials (with real coefficients) can be factored into linear and quadratic terms. Fundamental theorem of algebra

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. The graph of $y = \frac{1}{x^2}$ is graphed below:



(a) [2] Find the area trapped between $y = \dfrac{1}{x^2}$, the x-axis, from $x = 1$ to $x = 100$.

$$\underbrace{\int_1^{100}}_{.5} \underbrace{\frac{1}{x^2}\,dx}_{.5} = \underbrace{\left.\frac{-1}{x}\right]_1^{100}}_{.5} = \underbrace{\frac{-1}{100} - \frac{-1}{1}}_{.5} = 1 - \tfrac{1}{100} = \frac{99}{100} \text{ units}^2$$

(b) [1] Find $\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$ and interpret your answer in terms of area.

(.5) $\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left.\frac{-1}{x}\right]_1^b = \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1}\right) = \lim_{b\to\infty}\left(1 - \tfrac{1}{b}\right) = 1 - 0 = 1.$

(.5) The area trapped between $1/x^2$, the x-axis, & the the right of $x=1$ is finite and equal to one.

3. For each of the following outline the method(s) you would use to find the general antiderivative. Include in the descriptions which substitutions you would make and the fall out that would occur. Essentially, give the same level of detail as was given on the practice exam key.
For extra credit, find the general antiderivative. (Each correct and complete answer will earn 1%.)

try things (+.5)
sense/notation (+.5)

[3] (sorry-no meta-data for these)

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt$$

(+1) Substitution: let $u = t^2+2 \Rightarrow t^2 = u-2$
$\Rightarrow du = 2t\,dt \Rightarrow \frac{1}{2}du = t\,dt$

(+1) to get $\int \frac{t^2 t^2}{\sqrt{u}}\,t\,dt = \int \frac{(u-2)(u-2)}{u^{1/2}}\,du$
then foil + use prop. of exponents to simplify the expression to $\int u^{3/2} - 4u^{1/2} + 4u^{-1/2}\,du$ and integrate

or trig substitution

try things (+.5) sense (+.5) let $\sqrt{2}\tan u = t$
$\Rightarrow \sqrt{2}\sec^2 u\,du = dt$
Pythagorus $\Rightarrow$

(+1) $\int \frac{(\sqrt{2})^5 \tan^5 u \sqrt{2}\sec^2 u}{\sqrt{2}\sqrt{\tan^2 u+1}}\,du$

$\frac{2^3}{\sqrt{2}}\int \tan^5 u \sec u\,du$

finish with $w = \sec u$ & using pyth to turn the $\tan^2 u$ to sec's (.5) use Δ's, etc

[3] (sorry-no meta-data for these)

try things (+.5)
sense/notation (+.5)

$$\int \frac{x^3}{x^2+4x+3}\,dx$$

Use long division to write the above in the form (+1)

$\int$ polynomial + $\frac{\text{linear}}{x^2+4x+3}\,dx$

$= \int$ polynomial + $\frac{\text{linear}}{(x+1)(x+3)}\,dx$. (+

Use partial fraction to solve for A and B in

$\int$ polynomial + $\frac{A}{x+1} + \frac{B}{x+3}\,dx$,

(+.5) Then finish with substitution $u = x+1$, $w = x+3$ and natural logs

try things (+.5)
sense/notation (+.5)

[3] (sorry-no meta-data for these)

$$\int \tan(\theta)\sec^3(\theta)\,d\theta$$

Recall $\frac{d}{d\theta}(\sec\theta) = \tan\theta\sec\theta$

So the above is $\int \sec^2\theta \tan\theta\sec\theta\,d\theta$

(+1) Let $u = \sec\theta$
$\Rightarrow du = \sec\theta\tan\theta\,d\theta$

(+1) then we'd have $\int u^2\,du$
which we can integrate

try things (+.5)
sense/notation (+.5)

[3] (sorry-no meta-data for these)

$$\int v^3 e^{v^2}\,dv$$

(+1) Substitution: let $w = v^2 \Rightarrow dw = 2v\,dv$

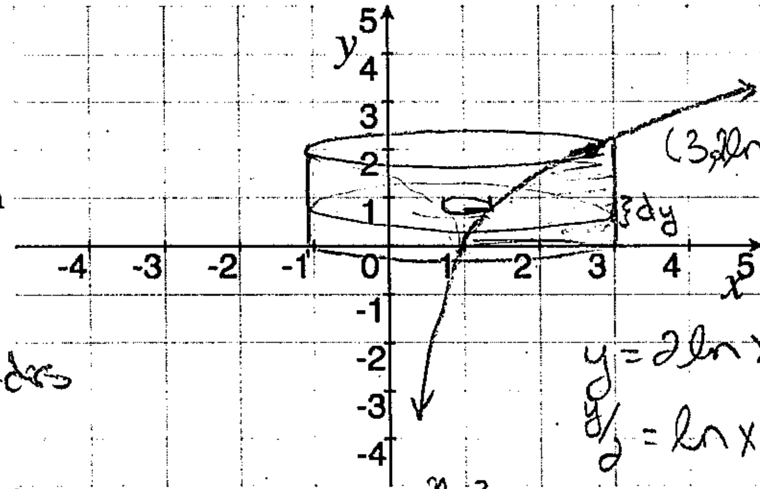
The above becomes
$\int v^2 e^w v\,dv = \int w e^w \frac{1}{2}\,dw$

(+1) $= \frac{1}{2}\int w e^w\,dw$.

Use integration by parts where
$u = w \qquad v = e^w$
$du = dw \qquad dv = e^w\,dw$

Then we'd have
$= \frac{1}{2}\left[ue^w - \int e^w\,dw\right]$
which we can integrate.

4. (§7.1 #64) Consider the region bounded by the curves $y = 2\ln(x)$, $y = 0$, and $x = 3$.



(a) [1] Carefully draw the region described above.

(b) [6] Find the volume that would result if the region was rotated about the line $x = 1$.

alg: (+.5)
notation/sense (+.5)

(+.5) { Vol of big cylinder − Vol of hole

$\pi R^2 h$ − Sum of approx cylinders (+.5)

$\pi\, 2^2 \ln 3$ − Sum $\pi r_I^2 \Delta y$

$\pi\, 4\ln 3$ − Sum $\pi(x\text{ coord}-1)^2\Delta y$ (+.5)

$\underbrace{4\pi\ln 3}_{(+.5)} - \int_{\underbrace{0}_{(+.5)}}^{2\ln 3} \underbrace{\pi(e^{y/2}-1)^2\,dy}_{(+1)}$

$y = 2\ln x$

$y/2 = \ln x$

$x = e^{y/2}$

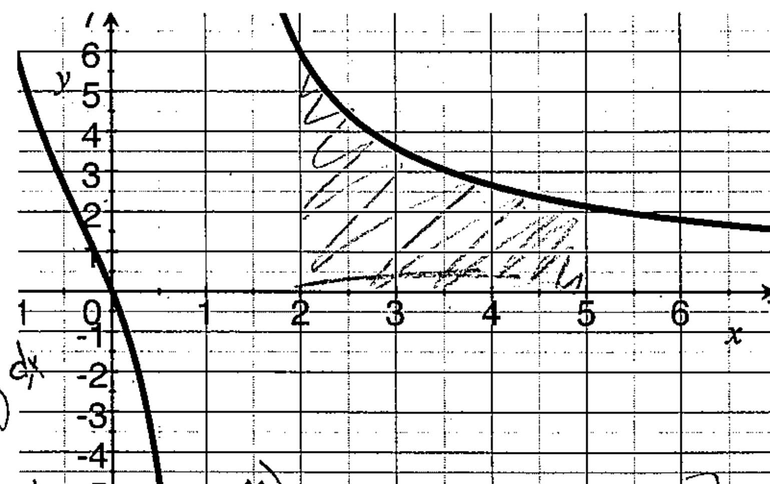
$4\pi\ln 3 - \pi\int_0^{2\ln 3} (e^{y/2})^2 - 2e^{y/2} + 1\,dy$

$4\pi\ln 3 - \pi\left[e^{y} - 4e^{y/2} + y\right]_0^{2\ln 3}$

$4\pi\ln 3 - \pi\left[(9 - 4\sqrt{3} + \ln 3) - (1 - 4 + 0)\right]$

integrate (+1)
FTC II (+.5)

5. (word wks #4) Consider $g(x) = \dfrac{12x}{x^2 + x - 2}$ graphed below.



(a) [6] Find the average value of g on the interval $[2, 5]$. Call your answer g_{ave}.

alg (+.5)
notation/sense (+.5)

$g_{ave}[5-2] = \int_2^5 g(x)\,dx$ } def (+1)

$g_{ave} = \frac{1}{5-2}\int_2^5 \frac{12x}{x^2+x-2}\,dx = \frac{1}{3}\int_2^5 \frac{12x}{(x+2)(x-1)}\,dx$

$= \frac{1}{3}\int_2^5 \frac{A}{x+2} + \frac{B}{x-1}\,dx \underset{\text{by } (*)}{=} \frac{1}{3}\int_2^5 \frac{8}{x+2} + \frac{4}{x-1}\,dx$

$= \frac{1}{3}\int_2^5 \frac{8}{x+2}\,dx + \frac{1}{3}\int_2^5 \frac{4}{x-1}\,dx = \underbrace{\frac{8}{3}\ln|x+2|}_{(+1)} + \underbrace{\frac{4}{3}\ln|x-1|}_{(+1)}\Big]_2^5$ plug (+.5)

$u = x+2$ $\quad w = x-1$

(*) alg: $\frac{A}{x+2} + \frac{B}{x-1} = \frac{12x}{(x+2)(x-1)}$ (+1.5) start +.5 setup +.5 got it +.5

$A(x-1) + B(x+2) = 12x$

$Ax + Bx - A + 2B = 12x + 0$

$\Rightarrow \begin{cases} A + B = 12 \\ -A + 2B = 0 \end{cases} \Rightarrow \begin{array}{l} 2B + B = 12 \Rightarrow B= \\ A = 2B \Rightarrow A = \end{array}$

(b) [2] Explain what $\int_2^5 g(x)\,dx$ and a 3 by g_{ave} rectangle have in common.

(+1) { a rectangle with dimensions 3 by g_{ave} has the same area as that shaded above in part (a). (+1)

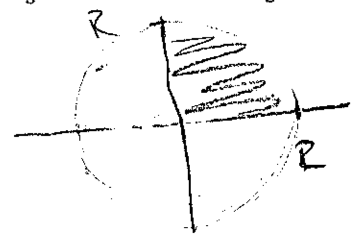
6. [6] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) (word wks #1) Recall that a circle is the collection of all (x, y) points a fixed distance r from a point (h, k). Use this and calculus to prove that the area of a circle with radius R is πR^2.

(b) (word wks #5) A rocket accelerates by burning its onboard fuel, so the mass of the rocket decreases with time. Suppose the initial mass of the rocket at lift off (including its fuel) is m, the fuel is consumed at a rate r, and the exhaust gases are ejected with constant velocity v_c (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_c ln\left(\frac{m - rt}{m}\right)$$

where g is the acceleration due to gravity and t is not too large. If the rocket is on mars where gravity is 3.69 m/s^2, $m = 30,000$kg, $r = 160\frac{\text{kg}}{\text{s}}$, and $v_c = 3000\frac{\text{m}}{\text{s}}$, find the height of the rocket one minute after liftoff.



(a) Consider the circle of radius R centered at the origin
(+.5) $\{x^2+y^2=R^2\}$. The area would be four times that of the shaded region or $\int_0^R \sqrt{R^2-x^2}\,dx$ (+.5) (+.5) (+.5)

(+1) $\begin{cases} x = R\sin\theta \\ du = R\cos\theta\, d\theta \end{cases}$

(+1) (+.5) $4\int_0^{\pi/2} \sqrt{R^2-(R\sin\theta)^2}\, R\cos\theta\, d\theta$

$4\int_0^{\pi/2} \sqrt{R^2(1-\sin^2\theta)}\, R\cos\theta\, d\theta = 4\int_0^{\pi/2} R\sqrt{1-\sin^2\theta}\, R\cos\theta\, d\theta$ (+.5)

b/c Pyth $1=\sin^2\theta+\cos^2\theta \Rightarrow 1-\sin^2\theta=\cos^2\theta$

$= 4R^2\int_0^{\pi/2} \cos\theta\cos\theta\, d\theta = 4R^2\int_0^{\pi/2}\cos^2\theta\, d\theta$

$\cos(2\theta)=\cos^2\theta-\sin^2\theta$
$\cos^2\theta=\cos(2\theta)+\sin^2\theta$
$=\cos(2\theta)+1-\cos^2\theta$
$\Rightarrow 2\cos^2\theta=\cos 2\theta+1$
$\cos^2\theta=\frac{1}{2}[\cos 2\theta+1]$

if $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ so dc

$= 4R^2\int_0^{\pi/2}\frac{1}{2}(\cos 2\theta+1)\,d\theta = 2R^2\int_0^{\pi/2}\cos 2\theta+1\,d\theta$

other integration tech (+1)

$= 2R^2\left[\frac{1}{2}\sin 2\theta+\theta\right]_0^{\pi/2} = 2R^2\left[\left(\frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right)+\frac{\pi}{2}\right)-\left(\frac{1}{2}\sin 0+0\right)\right]$

$= 2R^2\left[\frac{1}{2}\cdot 0+\frac{\pi}{2}-\frac{1}{2}\cdot 0\right] = 2R^2\cdot\frac{\pi}{2} = \pi R^2$:)

FTC II (+.5)

(b)

$$v(t) = \underbrace{-3.69 \cdot t}_{m/s} - \underbrace{3000 \cdot \ln\left(\frac{30{,}000 - 160 \cdot t}{30{,}000}\right)}_{m/s \quad 1} \Big\} \text{(+1)}$$

$\frac{m}{s^2}$ s $\frac{m}{s}$ $\frac{kg - \frac{kg}{s} s}{kg}$

velocity - units work.

find height when $t = 60s$

(+1) { Recall $\text{height}(t) = \int v(t)\,dt$ where we assume $v(0)=0$ ie the Rocket starts on the ground.

$$\text{height}(t) = \int -3.69t - 3000 \ln\left(\frac{3000 - 16t}{3000}\right) dt$$

other integration techniques (+1)

$$= -3.69 \cdot \tfrac{1}{2} t^2 - 3000 \int \ln\left(\frac{3000-16t}{3000}\right) dt$$

$w = \frac{3000 - 16t}{3000}$. $dw = \frac{-16}{3000}\,dt$

$$= -1.845t^2 - 3000 \cdot \frac{-3000}{16} \int \ln w \, dw$$

$u = \ln w$ $v = w$

IP (+1)

$du = 1/w$ $dv = dw$

$$= -1.845t^2 + \frac{9{,}000{,}000}{16}\left[w \ln w - \int w \cdot \tfrac{1}{w}\,dw\right]$$

$$= -1.845t^2 + \frac{9{,}000{,}000}{16}\left[\frac{3000-16t}{3000} \ln\left(\frac{3000-16t}{3000}\right) - \frac{3000-16t}{3000}\right] + c$$

$$= -1.845t^2 + \frac{3000}{16}\left[(3000-16t)\ln\left(\frac{3000-16t}{3000}\right) - 3000 + 16t\right] + c$$

and/or $\int_0^{60}$ limits (+1)

since $h(0)=0$ $\Rightarrow 0 = 0 + \frac{3000}{16}\left[(3000-0)\cancel{\ln(1)} - 3000 + 0\right] + c$

$$\Rightarrow 0 = \frac{3000}{16}(-3000) + c \Rightarrow c = \frac{9000000}{16}$$

alg/notation (+.5)

{ So height after 60 sec:

FTC (+.5)

$$-1.845(60)^2 + \frac{3000}{16}\left[(3000-16\cdot 60)\ln\left(\frac{3000-16\cdot 60}{3000}\right) - 3000 + 16(60)\right] + \frac{9000000}{16}$$

$$
\begin{array}{r}
2 \\
7 \\
12 \\
15 \\
6 \\
\hline
40
\end{array}
$$

Extra Credit Points.

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = \int \frac{t^2 t^2}{\sqrt{u}}\,t\,dt = \int \frac{(u-2)(u-2)}{u^{1/2}}\,\frac{1}{2}du = \frac{1}{2}\int \frac{u^2-4u+4}{u^{1/2}}\,du$$

$u = t^2+2 \Rightarrow u-2 = t^2$

$du = 2t\,dt \Rightarrow \frac{1}{2}du = t\,dt$

$$= \frac{1}{2}\int u^{3/2} - 4u^{1/2} + 4u^{-1/2}\,du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2} + 8u^{1/2}\right] + C$$

$$= \frac{1}{5}(t^2+2)^{5/2} - \frac{4}{3}(t^2+2)^{3/2} + 4(t^2+2)^{1/2} + C$$

Ck: $\left[\frac{1}{5}(t^2+2)^{5/2} - \frac{4}{3}(t^2+2)^{3/2} + 4(t^2+2)^{1/2} + C\right]'$

$$= \frac{1}{\cancel{2}}(t^2+2)^{3/2}\cancel{2}t - 2(t^2+2)^{1/2}2t + 2(t^2+2)^{-1/2}\cdot 2t + 0$$

$$= t\sqrt{t^2+2}^{\,3} - 4t\sqrt{t^2+2} + \frac{4t}{\sqrt{t^2+2}} = \frac{t(t^2+2)^2 - 4t(t^2+2) + 4t}{\sqrt{t^2+2}} = \frac{t^5 + \cancel{4t^3} + \cancel{4t} - \cancel{4t^3} - \cancel{8t} + \cancel{4t}}{\sqrt{t^2+2}}$$ ✓

$$\int \frac{x^3}{x^2+4x+3}\,dx = \int x - 4 + \frac{13x+12}{(x+3)(x+1)}\,dx = \int x - 4 + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{27}{2}}{x+3}\,dx$$

$$\begin{array}{r} x - 4 \\ x^2+4x+3 \overline{)\, x^3 + 0x^2 + 0x + 0} \\ -(x^3 + 4x^2 + 3x) \\ \hline -4x^2 - 3x + 0 \\ -(-4x^2 - 16x - 12) \\ \hline 13x + 12 \end{array}$$

$$\frac{A}{x+1} + \frac{B}{x+3} = \frac{13x+12}{(x+3)(x+1)}$$

$$Ax + 3A + Bx + B = 13x + 12$$

$$\begin{cases} 3A + B = 12 \\ A + B = 13 \end{cases} \overset{\text{sub}}{\Rightarrow} A = 13 - B$$

$3(13-B) + B = 12$

$39 - 2B = 12$

$-2B = -27$

$B = \frac{27}{2}$

$\Rightarrow A = 13 - \frac{27}{2} = \frac{-1}{2}$

$$\int \frac{x^3}{x^2+4x+3}\,dx = \frac{1}{2}x^2 - 4x - \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{27}{2}\int \frac{1}{x+3}\,dx$$

$u = x+1$ $\quad w = x+3$

$du = dx$ $\quad dw = dx$

$$= \frac{1}{2}x^2 - 4x - \frac{1}{2}\ln|x+1| + \frac{27}{2}\ln|x+3| + C$$

Ck: $\left[\frac{1}{2}x^2 - 4x - \frac{1}{2}\ln|x+1| + \frac{27}{2}\ln|x+3| + C\right]' = x - 4 + \frac{-\frac{1}{2}}{x+1} + \frac{\frac{27}{2}}{x+3} + 0$

$$= \frac{x(x+1)(x+3)}{(x+1)(x+3)} - \frac{4(x+1)(x+3)}{(x+1)(x+3)} + \frac{-\frac{1}{2}(x+3)}{(x+1)(x+3)} + \frac{\frac{27}{2}(x+1)}{(x+1)(x+3)} = \frac{x^3 + \cancel{4x^2} + \cancel{3x} - \cancel{4x^2} - \cancel{16x} - \cancel{12} - \cancel{\frac{1}{2}x} - \cancel{\frac{3}{2}} + \cancel{\frac{27}{2}x} + \cancel{\frac{27}{2}}}{(x+1)(x+3)}$$ ✓

$$\int \tan(\theta)\sec^3\theta\,d\theta = \int \sec^2\theta\,\tan\theta\sec\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + c$$

$u = \sec\theta$

$du = \tan\theta\sec\theta\,d\theta$

$$= \frac{1}{3}\sec^3\theta + c$$

Ck: $\left[\tfrac{1}{3}\sec^3\theta + c\right]' = \tfrac{1}{3}\cdot 3\sec^2\theta\,\sec\theta\tan\theta + 0 = \sec^3\theta\tan\theta$ ✓

$$\int v^3 e^{v^2}\,dv = \int v^2 e^{v^2}\,v\,dv = \int w e^{w}\,\tfrac{1}{2}\,dw = \tfrac{1}{2}\int w e^{w}\,dw$$

Sub

$w = v^2$

$dw = 2v\,dv \Rightarrow \tfrac{1}{2}\,dw = v\,dv$

IP

$u = w \qquad v = e^w$

$du = dw \qquad dv = e^w\,dw$

$$= \tfrac{1}{2}\left[we^w - \int e^w\,dw\right]$$

$$= \tfrac{1}{2}we^w - \tfrac{1}{2}e^w + c$$

$$= \tfrac{1}{2}v^2e^{v^2} - \tfrac{1}{2}e^{v^2} + c$$

Ck: $\left[\underbrace{\tfrac{1}{2}v^2e^{v^2}}_{\text{product rule}} - \tfrac{1}{2}e^{v^2} + c\right]' =$

$\left(\tfrac{1}{2}v^2\right)\left(e^{v^2}\right)' + \left(\tfrac{1}{2}v^2\right)'\left(e^{v^2}\right) - \cancel{\tfrac{1}{2}}e^{v^2}\cancel{2}v + 0$

$\cancel{\tfrac{1}{2}}v^2\cdot e^{v^2}\cancel{2}v + \cancel{\tfrac{1}{2}}\cancel{2}ve^{v^2} - e^{v^2}v$

$v^3e^{v^2} + \cancel{ve^{v^2}} - \cancel{e^{v^2}v} = v^3e^{v^2}$ ✓