Fx #2

EXAM 2 TMath 125 PRACTICE

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

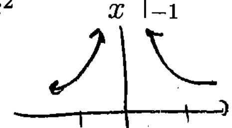
1. Each of the following is wrong. Explain why.

   (a) Substitution yields: $\int_0^1 y(y^2+1)^5 dy = \int_0^1 \frac{1}{2} u^5 du$

   The substitution is $u = y^2+1$. For the equal sign to be true we need to change the boundaries into terms of u? So from $0^2+1=1$ to $1^2+1=2$.

   (b) $\int_{-1}^{1} \frac{1}{x^2} dx = \frac{-1}{x}\Big|_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$

   note $y = \frac{1}{x^2}$



   The area should be positive if anything?
   The problem is that the FTC2 cannot be used on functions that are not continuous (like $y = 1/x^2$ @ $x=0$)

2. Technical communication questions:

   (a) How would you evaluate $\int \tan^a(x)\sec^b(x)\,dx$ if a is odd and b is even? And $a, b > 1$.

   Let $u = \tan(x)$. Then reserve $\sec^2(x)$ for the du ($du = \sec^2(x)dx$). Any remaining factors of $\sec(x)$ can be converted to $\tan(x)$'s with the Pythagorean theorem: $\sec^2(x) = 1+\tan^2(x)$. Since b is even, we can convert two $\sec(x)$'s at a time. We'll be left with

   $\int \tan^a(x)(1+\tan^2(x))^{b-2} \sec^2 x\, dx$. Then we can perform the substitution $= \int u^a (1+u^2)^{b-2} du$. Distribution will give us a polynomial to integrate with the power rule.

   (b) Provide a second way of evaluating $\int \tan^a(x)\sec^b(x)\,dx$ if a is odd and b is even. And $a, b > 1$.

   Let $u = \sec(x)$. Then reserve $\tan(x)$ and $\sec(x)$ for du ($du = \sec(x)\tan(x)$). Any remaining factors of $\tan(x)$ can be converted to $\sec(x)$'s with the Pythagorean theorem: $\tan^2(x) = \sec^2(x) - 1$.

   Since a is odd, we will have an even # of $\tan(x)$'s to convert & we can convert 2 at a time. We'll be left with

   $\int (\sec^2(x)-1)^{a-1} \sec^{b-1}(x) \sec(x)\tan(x)dx$. Then we perform the substitution

   $= \int (u^2-1)^{a-1} u^{b-1} du$. Distribution will give us a polynomial to integrate with the power rule.

3. For each of the following outline the method(s) you would use to find the general antiderivative. Choose *THREE* to evaluate. No, you do not get extra credit for evaluating more than one.

(a) $\int x4^x\,dx$

integration by parts

$u = x$

$dv = 4^x dx$

(b) $\int e^{\cos(t)} \sin(t)\cos(t)\,dt$

substitution $u = \cos(t)$ $du = -\sin(t)dt$

$\int e^u (-1) u\,du = -\int e^u u\,du$

integration by parts

$w = u$

$dv = e^u du$

(c) $\int 3\cot^3 x\,dx$

use Pythagorus to intro $\csc(x)$

$1 + \cot^2(x) = \csc^2(x)$

$\Rightarrow \int 3\cot(x)(\csc^2(x) - 1)dx$

split into 2 integrals w/ (algebra)

$\underbrace{\int 3\cot(x)\csc^2(x)dx}_{\text{sub } u=\cot(x)} - \underbrace{\int 3\cot(x)dx}_{\text{def of cot}}$

sub $u = \sin(x)$

(d) $\int \frac{dy}{(y^2+36)^{\frac{3}{2}}}$

trigonometric substitution

$y = 6\tan(\theta)$

$dy = 6\sec^2\theta\,d\theta$

use $\tan^2\theta + 1 = \sec^2\theta$

(e) $\int \frac{3+\sqrt{t}+t}{t}\,dt$

use algebra + simplify

$\int \frac{3}{t} + \frac{\sqrt{t}}{t} + \frac{t}{t}\,dt$

$\Rightarrow \int 3t^{-1} + t^{-1/2} + 1\,dt$

use the power rule (integral version)

(f) $\int \frac{(\ln(x))^2}{x^2}\,dx$

integration by parts (Twice?)

$u = (\ln(x))^2$

$dv = x^{-2}dx$

(e) $\displaystyle\int \frac{3+\sqrt{t}+t}{t}\,dt$ using algebra

$$= \int \frac{3}{t} + \frac{t^{1/2}}{t} + \frac{t}{t}\,dt$$

$$= \int {}^{3}/_{t}\,dt + \int t^{-1/2}\,dt + \int 1\,dt$$

$$= 3\ln|t| + 2t^{1/2} + t + C$$

Check: $\left[3\ln|t| + 2t^{1/2} + t + C\right]'$

$$= 3\,\frac{1}{t} + 2\cdot\frac{1}{2}\,t^{-1/2} + 1 + 0$$

$$= \frac{3}{t} + \frac{1}{\sqrt{t}} + 1 = \frac{3}{t} + \frac{\sqrt{t}}{t} + \frac{t}{t}$$

$$= \frac{3+\sqrt{t}+t}{t} \checkmark$$

(d) $\displaystyle\int \frac{dy}{(y^2+36)^{3/2}}$

trig substitution
$y = 6\tan\theta$
$dy = 6\sec^2\theta\,d\theta$

$$\int \frac{6\sec^2\theta}{\left[(6\tan\theta)^2+36\right]^{3/2}}\,d\theta = \int \frac{6\sec^2\theta\,d\theta}{36^{3/2}(\tan\theta+1)^{3/2}}$$

$$\left.\begin{array}{l}\text{note } \tan^2\theta+1 = \sec^2\theta \\ \Rightarrow (\tan^2\theta+1)^{1/2} = \sec\theta \\ \text{if } -90^\circ \le \theta \le 90^\circ\end{array}\right\} \int \frac{6\sec^2\theta\,d\theta}{6^3\sec^3\theta}$$

$$= \frac{1}{6^2}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{36}\int \cos\theta\,d\theta$$

$$= \frac{1}{36}\sin\theta + C$$

$$= \frac{1}{36}\,\frac{\text{opp}}{\text{hyp}} + C$$

$$= \frac{1}{36}\,\frac{y}{\sqrt{y^2+36}} + C$$

$y = 6\tan\theta$

${}^{y}/_{6} = \tan\theta$

Sohcahtoa

(f) $\displaystyle\int \frac{(\ln(x))^2}{x^2}\,dx$ integration by parts

$u = (\ln(x))^2$ $\qquad v = -x^{-1}$

$du = 2\ln(x)\cdot\frac{1}{x}\,dx$ $\qquad dv = x^{-2}\,dx$

$$= -(\ln(x))^2 x^{-1} - \int -x^{-1}\cdot 2\ln(x)\,\frac{1}{x}\,dx$$

$$= -\frac{(\ln(x))^2}{x} + 2\int \frac{\ln(x)}{x^2}\,dx$$

do integration by parts again? :)

$u = \ln(x)$ $\qquad v = -x^{-1}$

$du = \frac{1}{x}\,dx$ $\qquad dv = x^{-2}\,dy$

$$= -\frac{(\ln(x))^2}{x} + 2\left[-\ln(x)x^{-1} - \int -x^{-1}\frac{1}{x}\,dx\right]$$

$$= -\frac{(\ln(x))^2}{x} - \frac{2\ln(x)}{x} + 2\int \frac{1}{x^2}\,dy$$

$$= -\frac{(\ln(x))^2}{x} - 2\,\frac{\ln(x)}{x} - \frac{2}{x} + C$$

Check for (f)

$$\left[-\frac{(\ln(x))^2}{x} - \frac{2\ln(x)}{x} - \frac{2}{x} + C\right]'$$

$$\frac{+(\ln(x))^2}{x^2} + \frac{-1}{x}\,2\ln(x)\cdot\frac{1}{x} - \frac{-2\ln(x)}{x^2} - \cancel{\frac{2}{x\,x}}$$

$$+ \cancel{\frac{2}{x^2}} + 0$$

$$= \frac{(\ln(x))^2}{x^2} - \cancel{\frac{2\ln(x)}{x^2}} + \cancel{\frac{2\ln(x)}{x^2}} = \frac{(\ln(x))^2}{x^2} \checkmark$$

(a) $\int x 4^x dx$

integration by parts

$u = x \qquad v = \int 4^x \cdot \frac{1}{\ln 4}$

$du = dx \qquad dv = 4^x$

$x \frac{1}{\ln 4} \cdot 4^x - \int 4^x \frac{1}{\ln 4} dx$

$= \frac{x}{\ln 4} 4^x - \frac{1}{\ln 4} \int 4^x dx$

$= \frac{x}{\ln 4} \cdot 4^x - \frac{1}{\ln 4} \cdot \frac{1}{\ln 4} \cdot 4^x + C$

Check: $\left[\frac{x}{\ln 4} \cdot 4^x - \left(\frac{1}{\ln 4}\right)^2 \cdot 4^x + C\right]'$

$= \frac{x}{\cancel{\ln 4}} \cdot 4^x \cdot \cancel{\ln 4} + \frac{1}{\ln 4} \cdot 4^x - \left(\frac{1}{\ln 4}\right)^{\cancel{2}} \cdot 4^x \cancel{\ln 4}$

$= x 4^x + \cancel{\frac{4^x}{\ln 4}} - \cancel{\frac{4^x}{\ln 4}} = x 4^x$ ✓

(b) $\int e^{\cos(t)} \sin(t) \cos(t) dt$

Substitution $u = \cos(t)$

$du = -\sin(t) dt$

$\int e^u \cos(t) \sin(t) dt = \int e^u u (-1) du$

$= -\int e^u u du$ integration by parts

$w = u \qquad v = e^u$

$dw = du \qquad dv = e^u du$

$= -\left[u e^u - \int e^u du\right]$

$= -u e^u + \int e^u du = -u e^u + e^u + C$

$= -\cos(t) e^{\cos(t)} + e^{\cos(t)} + C$

Check: $\left[-\cos(t) e^{\cos(t)} + e^{\cos(t)} + C\right]'$

$= \cancel{\sin(t) e^{\cos(t)}} + (-\cos t) e^{\cos(t)}(-\sin(t))$

$\cancel{+ e^{\cos(t)}(-\sin t)} + 0$

$= e^{\cos(t)} \cos(t) \sin(t)$ ✓

(c) $\int 3 \cot^3 x dx = 3 \int \cot(x)(\csc^2(x) - 1) dx$

using Pythagoras $1 + \cot^2(x) = \csc^2(x)$

$3 \int \cot(x) \csc^2(x) dx - 3 \int \cot(x) dx$

$u = \cot(x)$

$du = -\csc^2(x) dx$

$= 3 \int u(-1) du - 3 \int \frac{\cos(x)}{\sin(x)} dx$

$w = \sin x$

$dw = \cos x\, dx$

$= -3 \frac{1}{2} u^2 - 3 \int \frac{1}{w} dw$

$= \frac{-3}{2} \cot^2(x) - 3 \ln|w| + C$

$= -\frac{3}{2} \cot^2(x) - 3 \ln|\sin(x)| + C$

Check: $\left[-\frac{3}{2} \cot^2(x) - 3 \ln|\sin(x)| + C\right]'$

$= \frac{-3}{2} \cdot 2 \cot(x)(-\csc^2(x)) - 3 \frac{1}{\sin x} \cdot \cos x$

$= 3 \cot(x) \csc^2(x) - 3 \cot(x)$

$= 3 \cot(x)(\csc^2(x) - 1)$

$= 3 \cot(x) \cot^2(x)$ ✓

4. Let $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and assume f" is continuous.

(a) Evaluate $\int_1^4 f'(x)\,dx$.

$$\int_1^4 f'(x)dx = f(x)\Big]_1^4 = f(4) - f(1) = 7 - 2 = 5$$

by Fundamental Theorem of Calc II   note $f(x)$ is an antiderivative of $f'(x)$

(b) Evaluate $\int_1^4 f''(x)\,dx$.

$$\int_1^4 f''(x)dx = f'(x)\Big]_1^4 = f'(4) - f'(1) = 3 - 5 = -2$$

by Fundamental Theorem of Calc II   note $f'(x)$ is an antiderivative of $f''(x)$

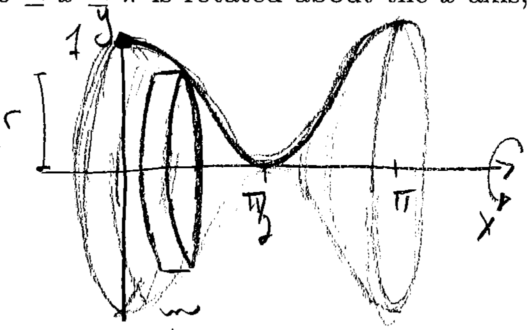
(c) Evaluate $\int_1^4 x f"(x)\,dx$.   use Integration by parts? $u = x$, $du = dx$ $v = f'(x)$, $dv = f''(x)dx$

$$= xf'(x)\Big]_1^4 - \int_1^4 f'(x)\,dx$$

$$= \underbrace{4f'(4) - 1\cdot f'(1)} - f(x)\Big]_1^4 = 4\cdot 3 - 5 - [f(4) - f(1)] = 7 - 5 = 2$$

5. The region under the curve $y = \cos^2(x)$ from $0 \le x \le \pi$ is rotated about the x-axis, find the volume of the resulting solid.

$\to$ VC1



Sum of approx cylinders

= Sum of (Area of circle) $\cdot$ (height of cyl)

= Sum of $(\pi r^2) \cdot \Delta x$

= Sum of $\pi(\text{ycoord})^2 \Delta x$

= Sum of $\pi(\cos^2(x))^2 \Delta x$

Take it to the limit:

$$\boxed{\int_0^\pi \pi(\cos^2(x))^2\,dx}$$

double angle identity $\pi \int \left(\frac{1}{2}\cos(2x) + \frac{1}{2}\right)^2 dx$

$$= \pi \int \frac{1}{4}\cos^2(2x) + \frac{1}{2}\cos(2x) + \frac{1}{4}\,dx$$

$$\frac{\pi}{4}\int \cos^2(2x)dx + \frac{\pi}{2}\int \cos(2x)dx + \pi\int \frac{1}{4}\,dx$$

$\downarrow$ double angle identity

$$\frac{\pi}{4}\int \frac{1}{2}\cos(2(2x)) + \frac{1}{2}\,dx + \frac{\pi}{2}\frac{1}{2}\sin(2x) + \frac{\pi}{4}x$$

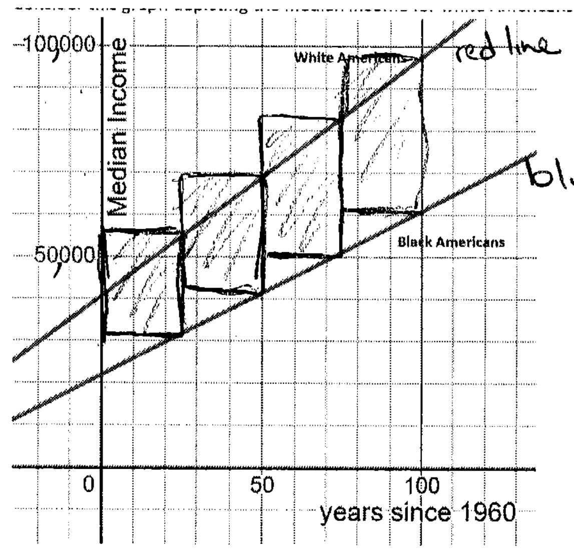
$$= \frac{\pi}{4}\left[\frac{1}{2}\frac{1}{4}\sin(4x) + \frac{1}{2}x\right] + \frac{\pi}{4}\sin(2x) + \frac{\pi}{4}x$$

$$= \frac{\pi}{32}\sin(4x) + \frac{\pi}{8}x + \frac{\pi}{4}\sin(2x) + \frac{\pi}{4}x + C$$

evaluate at π & 0

$$\left(0 + \frac{\pi^2}{8} + 0 + \frac{\pi^2}{4}\right) - (0+0+0+0) = \frac{3\pi^2}{8}$$

6. Consider the graphs depicting the median income for white Americans (top/red line) and Black Americans (bottom/blue line). The red line is described by the equation $y = 566x + 40,738$. The blue line is described by the equation $y = 390x + 21,970$.



(a) What is the meaning of the area between the red and blue lines from $x = 0$ to $x = 100$ (where x is years since 1960).

The total extra $ that the median White American has earned beyond the total $ earned by the median Black American

(b) Estimate the area identified in part a. Clearly indicate your process.

Approximate with 4 rectangles with right end points.

$$\text{Area} \approx (55{,}000-31{,}000)\cdot 25 + (70{,}000-42{,}000)\cdot 25 + (83{,}000-50{,}000)\cdot 25 + (98{,}000-60{,}000)\cdot 25$$

(c) Find the area between the curves from $x = 0$ to $x = 50$.

$$\int_0^{50} (566x + 40{,}738) - (390x + 21{,}970)\,dx = \int_0^{50} 176x + 18{,}768\,dx$$

$$= \frac{176}{2}x^2 + 18{,}768x \Big]_0^{50} = 1{,}158{,}400 - 0 = \$1{,}158{,}400 \approx 1.2\text{mil}$$

(d) Find the area between the curves from $x = 50$ to $x = 100$.

$$\int_{50}^{100} (566x + 40{,}738) - (390x + 21{,}970)\,dx = \int_{50}^{100} 176x + 18{,}768\,dx$$

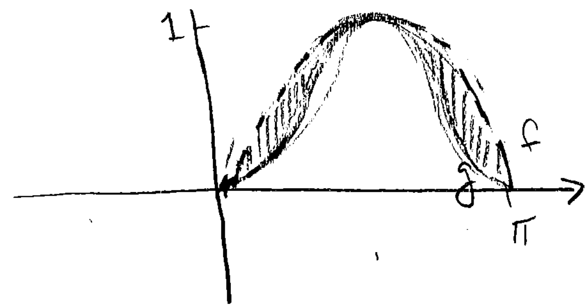
$$= \frac{176}{2}x^2 + 18{,}768x \Big]_{50}^{100} = 2{,}756{,}800 - 1{,}158{,}400 = \$1{,}598{,}400 \approx 1.6\text{mil}$$

(e) Is income inequality between the two races growing or shrinking? Justify your answer.

Growing. The median white american earned more ($1.6mil) money in the past 50 years than the previous 50 years ($1.2 mil) beyond the median Black American.

7. Let $f(x) = \sin(x)$, $g(x) = \sin^3(x)$.

(a) Sketch a graph of f and g.



(b) Find the area trapped between the graph of f and g from $x = 0$ to $x = \pi$.

$$\int_0^{\pi} \sin(x) - \sin^3(x)\,dx = \int_0^{\pi} \sin(x)\,dx - \int_0^{\pi} \sin^2(x)\sin(x)\,dx$$

use $\sin^2(x) = 1 - \cos^2(x)$
and $u = \cos(x)$
$du = -\sin(x)\,dx$

$$= -\cos(x)\Big]_0^{\pi} - \int_0^{\pi} (1-\cos^2(x))\sin(x)\,dx$$

$$= -\cos(\pi) + \cos(0) + \int^{-1} 1 - u^2\,du$$

$$= +1 + 1 + \left[u - \tfrac{1}{3}u^3\right]_1^{-1}$$

$$= 2 + \left[-1 + \tfrac{1}{3}\right] - \left[1 - \tfrac{1}{3}\right] = 2 + \left[\tfrac{-2}{3} - \tfrac{2}{3}\right] = 2 - \tfrac{4}{3} = \tfrac{2}{3}$$

(c) Consider the object whose base is bounded by f, the x axis, $x = 0$ and $x = \pi$. The cross sections (perpendicular to the x axis) of the object are squares. Set up the integral to find the volume of this object.

Sum of approx cylinders

= sum of (Area of square) $\cdot$ (height of cyl)

= sum of (side length)$^2 \cdot \Delta x$

= sum of $(f-g)^2\,\Delta x$

= sum of $[f(x) - g(x)]^2\,\Delta x$

Take it to the limit:

$$\int_0^{\pi} [\sin(x) - \sin^3(x)]^2\,dx$$