Ave 75% } :)
Median 79% }

Key

Exam 2

TMath 125

Autumn 2023

Expression	Substitution	Restrictions	Reason
$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2(\theta) = \cos^2(\theta)$
	OR $x = a\cos(\theta)$	$0 \le \theta \le \pi$	$1-\cos^2(\theta) = \sin^2(\theta)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta)$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$	$1+\tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$	$0 \le \theta \le \pi$	$\sec^2(\theta)-1 = \tan^2(\theta)$

Reasonable supporting work must be shown to earn credit.

1. Each of the following is wrong. Explain why.

(a) [2] (10/23 Activity #1) $\int 2x\sin(x)\,dx = -x^2\cos(x) + c$

true/did not miss it (.5)
sense (.5)
IP vs mult (1)

The integral of the product is not the product of the integrals!
$\int 2x\sin(x)dx \neq \int 2x\,dx \cdot \int \sin(x)dx$.
Instead we have to use integration by parts

(b) [2] (WebHW7-3 #2) We let $x = \sqrt{7}\sec(\theta)$, then $\int \frac{x^2}{\sqrt{x^2-7}}\,dx = \frac{7\sec^2(\theta)}{\sqrt{7\sec^2(\theta)-7}} \cdot \tan(\theta)\sec(\theta)$

$dx = \sqrt{7}\sec\theta\tan\theta\,d\theta$

true/did not miss TD (.5)
sense (.5)
Lost integral (1)

We lost the integral when doing the trig substitution!
We still have to integrate!

(c) [2] (PracticeExam#1) Substitution $\int_0^1 y(y^2+1)^5\,dy = \int_0^1 \frac{1}{2}u^5\,du$

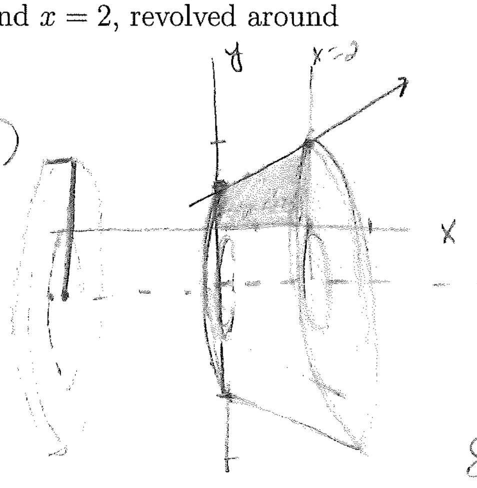
true/did not miss SD (.5)
sense (.5)
bounds (1)

We let $y^2+1 = u$ and did a substitution.
Either drop the bounds or change them to be with respect to u

(d) [2] The region bounded by $y = \frac{1}{2}x+1$, $y=0$, $x=0$, and $x=2$, revolved around the line $y=-1$ has volume $\int_0^2 \pi\left(\frac{1}{2}x+1\right)^2\,dx$

true/did not miss SD
sense (.5)
radius/approx cyl (1)

The approx disks are not solid like
there is a hole in them!
Also the radius is not the y-coord
but r_{outer} = ycoord + 1 = $\left(\frac{1}{2}x+1\right)+1$



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2. [4] (Quiz4 #1) Describe the strategy for evaluating $\int \cot^m(x)\csc^n(x)\,dx$ when m is odd and both $m, n > 0$. Consider the two worked out examples on the right.

Start (1.5)
generalizing (1.5)

(+1) Use substitution with $u = \csc(x)$

(+1) Reserve one $\cot(x)$ factor and one $\csc(x)$ factor for du

(+1) Take any remaining (even) powered $\cot^2(x)$ and swap in $\csc^2 x - 1$ to finish getting in terms of u

$\int \cot^3(x)\csc(x)\,dx$

$= \int \cot^2(x)\,\underline{\cot(x)\csc(x)\,dx}$

$u = \csc(x)$

$du = (\csc(x))' = \left(\frac{1}{\sin x}\right)' dx$

$= \frac{\sin(x)\cdot 0 - 1\cos x}{\sin^2 x}\,dx$

$= -\frac{\cos x}{\sin x}\cdot\frac{1}{\sin x}\,dx$

$= -\cot(x)\csc(x)\,dx$

$\Rightarrow -du = \cot(x)\csc(x)\,dx$

Recall $\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$\Rightarrow 1 + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$

$\Rightarrow 1 + \cot^2 x = \csc^2 x$

$\Rightarrow \cot^2(x) = \csc^2 x - 1$

$\int \cot^2(x)(-du) = \int (\csc^2 x - 1)(-du)$

$= -\int u^2 - 1\,du = -\left(\frac{1}{3}u^3 - u\right) + C$

$= \frac{-1}{3}\csc^3(x) + \csc(x) + C$

CK: $\left[-\frac{1}{3}\csc^3(x) + \csc(x) + C\right]' = \frac{+3}{3}\csc^2(x)(+\cot(x)\csc(x)) - \csc(x)\cot(x)$ ✓

$\int \cot(x)\csc^3(x)\,dx$

$= \int \csc^2(x)\,\cot(x)\csc(x)\,dx$

$u = \csc(x)$

$du = -\cot(x)\csc(x)\,dx$

$\Rightarrow -du = \cot(x)\csc(x)\,dx$

$\int u^2(-du) = -\int u^2\,du$

$= -\frac{1}{3}u^3 + C$

$= -\frac{1}{3}\csc^3(x) + C$

CK: $\left[-\frac{1}{3}\csc^3(x) + C\right]'$

$= \frac{+1}{\cancel{3}}\cancel{3}\csc^2(x)(\cot(x)\csc(x))$ ✓

3. For each of the following, identify the technique you would use to find the indefinite integral. For example, if you think substitution would work, write "substitution" and identify what u would be. If you think integration by parts, write "integration by parts" and identify what u and dv would be.

Start (.5)
method (.5)
works (1)

(a) [2]

$$\int 4\sin^2(t)\cos^3(t)\,dt$$

$u = \sin(t)$
$du = \cos(t)\,dt$
switch remaining $\cos^2(t)$ to $1 - \sin^2(t)$

Start (.5)
method (.5)
works (1)

(b) [2]

$$\int \frac{y}{\sqrt{4y^2+9}}\,dy$$

$u = 4y^2 + 9$
$du = 8y\,dy$

OR Trig sub?
$2y = 3\tan\theta$
$y = \frac{3}{2}\tan\theta$

4. [4] Evaluate *one* of the indefinite integrals above.

$\int 4\sin^2(t)\cos^3(t)\,dt$

$u = \sin(t)$
$du = \cos(t)\,dt$

$= 4\int u^2\cos^2(t)\,du$

recall
$\sin^2 t + \cos^2 t = 1$
$\Rightarrow \cos^2 t = 1 - \sin^2 t$

$= 4\int u^2(1 - \sin^2(t))\,du$

$= 4\int u^2(1 - u^2)\,du$

$= 4\int u^2 - u^4\,du$

$= 4\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = \frac{4}{3}\sin^3(t) - \frac{4}{5}\sin^5(t) + C$

OR

$\int \frac{y}{\sqrt{4y^2+9}}\,dy$

$u = 4y^2 + 9$
$du = 8y\,dy$
$\frac{1}{8}du = y\,dy$

$= \int \frac{\frac{1}{8}du}{\sqrt{u}}$

$= \frac{1}{8}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{8}2u^{\frac{1}{2}} + C$

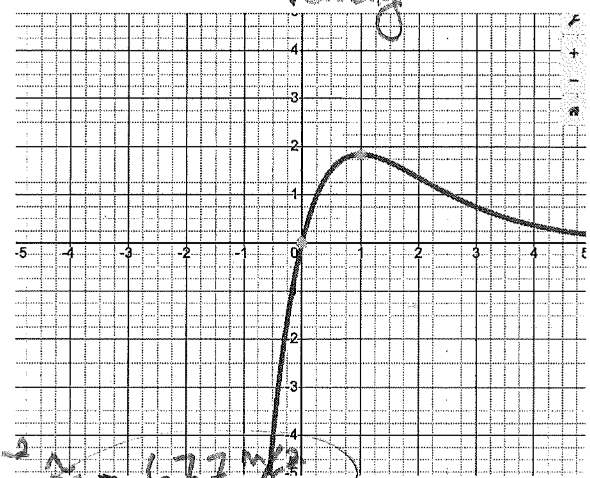
$= \frac{1}{4}\sqrt{4y^2+9} + C$

notation (.5)
Start (1.5)
tried something correctly (+2)
ic (.5)
got it (.5)

CK: $\left[\frac{1}{4}[4y^2+9]^{\frac{1}{2}}\right]'$
$= \frac{1}{4}\cdot\frac{1}{2}(4y^2+9)^{-\frac{1}{2}}\cdot 8y$ ✓ :)

5. A particle is moving along a straight line and has a velocity $v(t) = 5te^{-t}$ meters per second after t seconds.

velocity



(a) [1] (Quiz4#3) Find the velocity of the particle when $t = 2$.

$v(2) = 5(2)e^{-2} \approx 1.35 \text{ m/s}$

(b) [3] (WordProblem2#1) Find the acceleration of the particle when $t = 2$.

start (.5)
notation (.5)

(+1) [acceleration $= \frac{d}{dt}(v(t))$
$= (5t)e^{-t}(-1) + 5 \cdot e^{-t}$

(+1)
$\Rightarrow$ @ $t=2$ is $-5(2)e^{(2)} + 5e^{-2} \approx -.677 \text{ m/s}^2$

(c) [3] (WrittenHW7.1#75) Find the expression that you could give to technology that would return the change in distance in the first 2 seconds.

notation (.5)
bounds (+1)
int (.5)
v(t) function (.5)
got it (.5)

distance $= \int v(t)dt$

dist in first 2 seconds $= \int_0^2 5te^{-t}\,dt \approx 2.97\text{m}$

6. Let f be a twice differentiable function with the values f and f' given below.

x	$f(x)$	$f'(x)$
1	-2	4
5	3	5

(a) [1] Find $f(5)$, if possible.

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start (.5)
notation (.5)
antider (+1)
plug in values (+1)

(b) [3] (PracticeExam2#4) Evaluate $\int_1^5 f''(x)\,dx$, if possible.

note f' is an antiderivative of $f''(x)$ so

FTC $\Rightarrow \int_1^5 f''(x)dx = f'(x)\Big]_1^5 = f'(5) - f'(1) = 5 - 4 = 1$

(c) [3] (Quiz4#2) Evaluate $\int_1^5 6xf''(x)\,dx$, if possible.

try method correctly (1.5) IP
notation (.5)
plug in values (+1)

$u = 6x \qquad v = f'(x)$

$du = 6dx \qquad dv = f''(x)dx$

$\int_1^5 6xf''(x)dx = 6x\,f'(x)\Big]_1^5 - \int_1^5 f'(x)\,6dx$

$= [6 \cdot 5f'(5) - 6 \cdot 1 \cdot f'(1)] - 6f(x)\Big]_1^5 = (30 \cdot 5 - 6 \cdot 4) - (6(f(5) - f(1)))$

$= 150 - 24 - 6(3+2) = 126 - 30 = 96$

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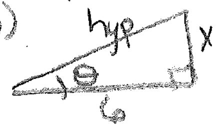
30
50 Total

7. [3] One problem substituted $x = 6\tan(\theta)$ and then integrated to get $\frac{1}{36}\sin(\theta) + C$. Convert this answer (that is currently in terms of θ) back into terms of x.

sketch (.5)
triangle/trig SD (+1.5)
alg/pyth (+1)

$\frac{x}{6} = \frac{6\tan(\Theta)}{6}$



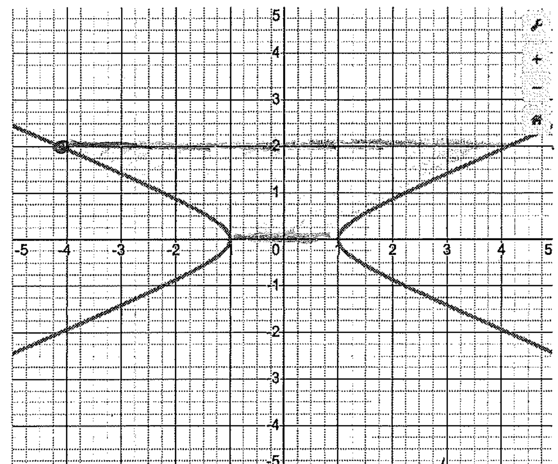
$\Rightarrow hyp = \sqrt{x^2+36}$

$\tan\Theta = \frac{x}{6}$ Sohcahtoa

So $\frac{1}{36}\sin\Theta + C$

$= \frac{1}{36}\frac{x}{\sqrt{x^2+36}} + C$

8. The graph of $x^2 - 4y^2 = 1$ is given below.



(a) [2] Shade the region trapped by $x^2 - 4y^2 = 1$, $y = 2$, and the x-axis.

(.5) (+.5) (.5)

got the area (.5)

(b) [4] (WrittenHW§7.3#40)
Set up the definite integral (but do *not* compute!) that will find the area of the shaded region. $x^2 = 1 + 4y^2$

integral (+.5)
bounds (+1)
notation (+.5)
Connect to x coord (.5)
connect to y coord (+1)
got it (.5)

$\int_0^2 (\text{bigger } x \text{ value}) - (\text{smaller } x \text{ value})\, dy$

$\int_0^2 \sqrt{1+4y^2} - -\sqrt{1+4y^2}\, dy$

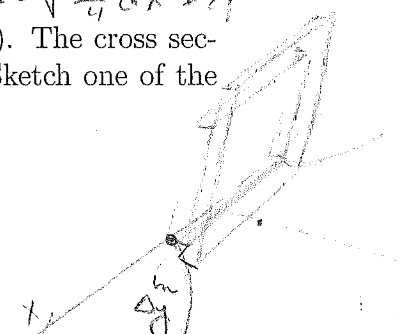
OR with respect to x (3 regions)

$\int_{-4.123}^{-1} 2 - y\text{ value}\, dx + 2\cdot 2 + \int_{1}^{4.123} 2 - y\text{ value}\, dx$

$\Rightarrow 2\int_{-4.123}^{-1} 2 - \sqrt{\frac{x^2-1}{4}}\, dx + 4$

(c) [2] (§6.2#2) Consider the volume whose base was shaded in (a). The cross sections (perpendicular to the y axis) of the object are squares. Sketch one of the approximating cylinders of this volume.

x,y,z axis (.5)
square cylinders (+1)
oriented right (.5)



(d) [4] (WordProblem2#7) Set up the definite integral that would find the volume described above.

integral (.5)
bounds (+1)
Δy (.5)
Connect to x coord (.5)
connect to y (.5)
squared (.5)

$\int_0^2 (\text{length of square})^2 \cdot dy$

$\int_0^2 (2x\text{-coord})^2\, dy$

$\int_0^2 \left(2\sqrt{1+4y^2}\right)^2 dy$ got it (.5)

9. [1] Provide a topic or problem that you studied for but did not see on the exam.

(+1) Something from the class

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