3.8 Implicit Differentiation

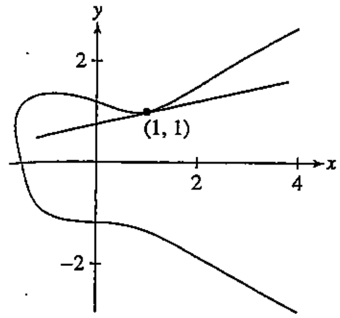

FIGURE 1 Graph of the implicitly defined function $y^4 + xy = x^3 - x + 2$.

We have developed the basic techniques for calculating a derivative dy/dx when y is given in terms of x by a formula—such as $y = x^3 + 1$. But suppose that y is determined instead by an equation such as

$$y^4 + xy = x^3 - x + 2 \qquad \boxed{1}$$

In this case, we say that y is defined *implicitly*. How can we find the slope of the tangent line at a point on the graph (Figure 1)? Although it may be difficult or even impossible to solve for y explicitly as a function of x, we can find dy/dx using the method of **implicit differentiation.**

To illustrate, consider the equation of the unit circle (Figure 2):

$$x^2 + y^2 = 1$$

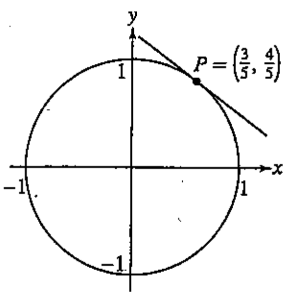

FIGURE 2 The tangent line to the unit circle $x^2 + y^2 = 1$ at P has slope $-\frac{3}{4}$.

Compute dy/dx by taking the derivative of both sides of the equation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + \frac{d}{dx}(y^2) = 0 \qquad \boxed{2}$$

How do we handle the term $\frac{d}{dx}(y^2)$? We use the Chain Rule. Think of y as a function $y = f(x)$. Then $y^2 = (f(x))^2$ and by the Chain Rule,

$$\frac{d}{dx}y^2 = \frac{d}{dx}(f(x))^2 = 2f(x)\frac{df}{dx} = 2y\frac{dy}{dx}$$

Equation (2) becomes $2x + 2y\frac{dy}{dx} = 0$, and we can solve for $\frac{dy}{dx}$ if $y \neq 0$:

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}} \qquad \boxed{3}$$

■ **EXAMPLE 1** Use Eq. (3) to find the slope of the tangent line at the point $P = \left(\frac{3}{5}, \frac{4}{5}\right)$ on the unit circle.

Solution Set $x = \frac{3}{5}$ and $y = \frac{4}{5}$ in Eq. (3):

$$\left.\frac{dy}{dx}\right|_P = -\frac{x}{y} = -\frac{\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$$ ■

In this particular example, we could have computed dy/dx directly, without implicit differentiation. The upper semicircle is the graph of $y = \sqrt{1 - x^2}$ and

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{1 - x^2} = \frac{1}{2}\left(1 - x^2\right)^{-1/2}\frac{d}{dx}\left(1 - x^2\right) = -\frac{x}{\sqrt{1 - x^2}}$$

This formula expresses dy/dx in terms of x alone, whereas Eq. (3) expresses dy/dx in terms of both x and y, as is typical when we use implicit differentiation. The two formulas agree because $y = \sqrt{1 - x^2}$.

Before presenting additional examples, let's examine again how the factor dy/dx arises when we differentiate an expression involving y with respect to x. It would not appear if we were differentiating with respect to y. Thus,

$$\frac{d}{dy}\sin y = \cos y \quad \text{but} \quad \frac{d}{dx}\sin y = (\cos y)\frac{dy}{dx}$$

$$\frac{d}{dy}y^4 = 4y^3 \quad \text{but} \quad \frac{d}{dx}y^4 = 4y^3\frac{dy}{dx}$$

Notice what happens if we insist on applying the Chain Rule to $\frac{d}{dy}\sin y$. The extra factor appears, but it is equal to 1:

$$\frac{d}{dy}\sin y = (\cos y)\frac{dy}{dy} = \cos y$$

Similarly, the Product Rule applied to xy yields

$$\frac{d}{dx}(xy) = \frac{dx}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}$$

The Quotient Rule applied to t^2/y yields

$$\frac{d}{dt}\left(\frac{t^2}{y}\right) = \frac{y\frac{d}{dt}t^2 - t^2\frac{dy}{dt}}{y^2} = \frac{2ty - t^2\frac{dy}{dt}}{y^2}$$

■ **EXAMPLE 2** Find an equation of the tangent line at the point $P = (1, 1)$ on the curve (Figure 1)

$$y^4 + xy = x^3 - x + 2$$

Solution We break up the calculation into two steps.

Step 1. **Differentiate both sides of the equation with respect to x.**
Note that each occurrence of y in the original equation generates an additional $\frac{dy}{dx}$ upon differentiation.

$$\frac{d}{dx}y^4 + \frac{d}{dx}(xy) = \frac{d}{dx}(x^3 - x + 2)$$

$$4y^3\frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 3x^2 - 1 \qquad \boxed{4}$$

Step 2. **Solve for $\frac{dy}{dx}$.**

Move the terms involving dy/dx in Eq. (4) to the left and place the remaining terms on the right:

$$4y^3\frac{dy}{dx} + x\frac{dy}{dx} = 3x^2 - 1 - y$$

Then factor out dy/dx and divide:

$$(4y^3 + x)\frac{dy}{dx} = 3x^2 - 1 - y$$

$$\frac{dy}{dx} = \frac{3x^2 - 1 - y}{4y^3 + x} \qquad \boxed{5}$$

To find the derivative at $P = (1, 1)$, apply Eq. (5) with $x = 1$ and $y = 1$:

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3 \cdot 1^2 - 1 - 1}{4 \cdot 1^3 + 1} = \frac{1}{5}$$

An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$. ■

CONCEPTUAL INSIGHT The graph of an equation does not always define a function because there may be more than one y-value for a given value of x. Implicit differentiation works because the graph is generally made up of several pieces called **branches**, each of which does define a function (a proof of this fact relies on the Implicit Function Theorem from advanced calculus). For example, the branches of the unit circle $x^2 + y^2 = 1$ are the graphs of the functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$. Similarly, the graph in Figure 3 has an upper and a lower branch. In most examples, the branches are differentiable except at certain exceptional points where the tangent line may be vertical.

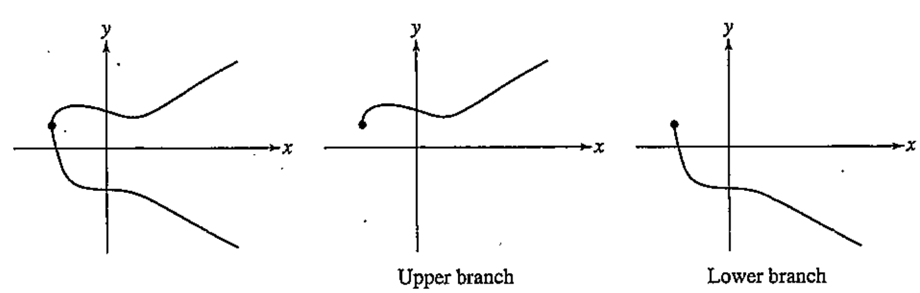

FIGURE 3 Each branch of the graph of $y^4 + xy = x^3 - x + 2$ defines a function of x.

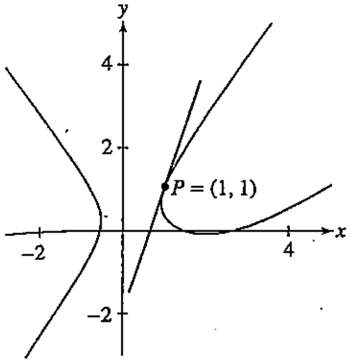

FIGURE 4 Graph of $e^{x-y} = 2x^2 - y^2$.

■ **EXAMPLE 3** Find the slope of the tangent line at the point $P = (1, 1)$ on the graph of $e^{x-y} = 2x^2 - y^2$.

Solution We follow the steps of the previous example, this time writing y' for dy/dx:

$$\frac{d}{dx}e^{x-y} = \frac{d}{dx}(2x^2 - y^2)$$

$$e^{x-y}(1 - y') = 4x - 2yy' \qquad \text{(Chain Rule applied to } e^{x-y}\text{)}$$

$$e^{x-y} - e^{x-y}y' = 4x - 2yy'$$

$$(2y - e^{x-y})y' = 4x - e^{x-y} \qquad \text{(place all } y'\text{-terms on left)}$$

$$y' = \frac{4x - e^{x-y}}{2y - e^{x-y}}$$

The slope of the tangent line at $P = (1, 1)$ is (Figure 4)

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{4(1) - e^{1-1}}{2(1) - e^{1-1}} = \frac{4-1}{2-1} = 3$$ ■

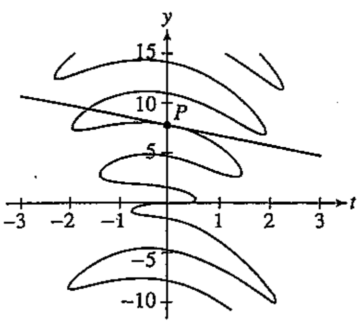

FIGURE 5 Graph of $y\cos(y + t + t^2) = t^3$. The tangent line at $P = \left(0, \frac{5\pi}{2}\right)$ has slope -1.

■ **EXAMPLE 4 Shortcut to Derivative at a Specific Point** Calculate $\left.\frac{dy}{dt}\right|_P$ at the point $P = \left(0, \frac{5\pi}{2}\right)$ on the curve (Figure 5):

$$y\cos(y + t + t^2) = t^3$$

Solution As before, differentiate both sides of the equation (we write y' for dy/dt):

$$\frac{d}{dt}y\cos(y + t + t^2) = \frac{d}{dt}t^3$$

$$y'\cos(y + t + t^2) - y\sin(y + t + t^2)(y' + 1 + 2t) = 3t^2 \qquad \boxed{6}$$

We could continue to solve for y', but that is not necessary. Instead, we can substitute $t = 0$, $y = \frac{5\pi}{2}$ directly in Eq. (6) to obtain

$$y'\cos\left(\frac{5\pi}{2} + 0 + 0^2\right) - \left(\frac{5\pi}{2}\right)\sin\left(\frac{5\pi}{2} + 0 + 0^2\right)(y' + 1 + 0) = 0$$

$$0 - \left(\frac{5\pi}{2}\right)(1)(y' + 1) = 0$$

This gives us $y' + 1 = 0$ or $y' = -1$. ■

Derivatives of Inverse Trigonometric Functions

We now apply implicit differentiation to determine the derivatives of the inverse trigonometric functions. An interesting feature of these functions is that their derivatives are not trigonometric. Rather, they involve quadratic expressions and their square roots. Keep in mind the restricted domains of these functions.

THEOREM 1 Derivatives of Arcsine and Arccosine

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \boxed{7}$$

Proof If $y = \sin^{-1} x$, our goal is to find $\frac{dy}{dx}$. By applying sine to both sides, we have

$$\sin y = x$$

Differentiating both sides of the equation, treating x as itself and y as a function of x, we obtain

$$\cos y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$

In order to determine an algebraic expression in x for $\cos y$, we construct a right triangle as in Figure 6 such that $\sin y = x$. We choose y to be its angle, and take its hypotenuse to be of length 1 and its opposite edge to have length x. Then, by the Pythagorean Theorem, its adjacent side must have length $\sqrt{1-x^2}$. We can therefore read off the triangle that $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Thus,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

The fact that the domain of the inverse sine function is from $-\pi/2$ to $\pi/2$, over which cosine is nonnegative, allows us to take the positive square root rather than the negative square root.

The computation of $\frac{d}{dx}(\cos^{-1} x)$ is similar (see Exercise 45 or the next example). ∎

REMINDER In Example 7 of Section 1.5, we used the right triangle in Figure 6 in the computation:

$$\cos(\sin^{-1} x) = \cos y = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1-x^2}$$

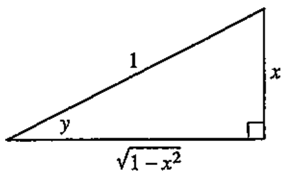

FIGURE 6 Right triangle constructed so that $\sin y = x$.

■ **EXAMPLE 5 Complementary Angles** The derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are equal up to a minus sign. Explain this by proving that

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Solution In Figure 7, we have $\theta = \sin^{-1} x$ and $\psi = \cos^{-1} x$. These angles are complementary, so $\theta + \psi = \frac{\pi}{2}$ as claimed. Therefore,

$$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x$$
$$\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\left(\frac{\pi}{2} - \cos^{-1} x\right) = -\frac{d}{dx}\cos^{-1} x$$ ∎

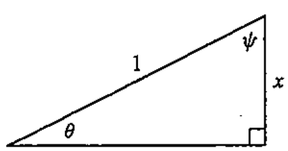

FIGURE 7 The angles $\theta = \sin^{-1} x$ and $\psi = \cos^{-1} x$ are complementary and thus sum to $\pi/2$.

■ **EXAMPLE 6** Calculate $f'\left(\frac{1}{2}\right)$, where $f(x) = \arcsin(x^2)$.

Solution Recall that $\arcsin x$ is another notation for $\sin^{-1} x$. By the Chain Rule,

$$\frac{d}{dx}\arcsin(x^2) = \frac{d}{dx}(\sin^{-1}(x^2)) = \frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$
$$f'\left(\frac{1}{2}\right) = \frac{2\left(\frac{1}{2}\right)}{\sqrt{1-\left(\frac{1}{2}\right)^4}} = \frac{1}{\sqrt{\frac{15}{16}}} = \frac{4}{\sqrt{15}}$$ ∎

THEOREM 2 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{x^2+1}, \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{x^2+1}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$$

The proofs of the formulas in Theorem 2 are similar to the proof of Theorem 1. See Exercises 46–48.

EXAMPLE 7 Calculate $\left.\frac{d}{dx}(\csc^{-1}(e^x+1))\right|_{x=0}$.

Solution Apply the Chain Rule using the formula $\frac{d}{du}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}$:

$$\frac{d}{dx}\csc^{-1}(e^x+1) = -\frac{1}{|e^x+1|\sqrt{(e^x+1)^2-1}}\frac{d}{dx}(e^x+1)$$
$$= -\frac{e^x}{(e^x+1)\sqrt{e^{2x}+2e^x}}$$

We have replaced $|e^x+1|$ by e^x+1 because this quantity is positive. Now we have

$$\left.\frac{d}{dx}\csc^{-1}(e^x+1)\right|_{x=0} = -\frac{e^0}{(e^0+1)\sqrt{e^0+2e^0}} = -\frac{1}{2\sqrt{3}}$$ ∎

Finding Higher Order Derivatives Implicitly

We may need to find a higher order derivative of a function that is defined implicitly, as in the next example.

EXAMPLE 8 Find a formula for $\frac{d^2y}{dx^2}$ if y is defined implicitly as a function of x by $x^2+4y^2=7$.

Solution We differentiate with respect to x, writing y' for $\frac{dy}{dx}$.

$$2x+8yy'=0$$

Solving for y', we obtain

$$y' = \frac{-x}{4y}$$

Differentiating again with respect to x, we obtain

$$y'' = \frac{4y(-1)-(-x)(4y')}{16y^2} = \frac{-y+xy'}{4y^2}$$

Substituting in the fact that $y' = \frac{-x}{4y}$ yields

$$y'' = \frac{-y+x(-x/4y)}{4y^2} = \frac{-4y^2-x^2}{16y^3}$$ ∎

3.8 SUMMARY

- Implicit differentiation is used to compute dy/dx when x and y are related by an equation.

  Step 1. Take the derivative of both sides of the equation with respect to x.
  Step 2. Solve for dy/dx by collecting the terms involving dy/dx on one side and the remaining terms on the other side of the equation.

- Remember to include the factor dy/dx when differentiating expressions involving y with respect to x. For instance,

$$\frac{d}{dx}\sin y = (\cos y)\frac{dy}{dx}$$

- Derivative formulas:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

3.8 EXERCISES

Preliminary Questions

1. Which differentiation rule is used to show $\frac{d}{dx}\sin y = \cos y \frac{dy}{dx}$?

2. One of (a)–(c) is incorrect. Find and correct the mistake.

(a) $\frac{d}{dy}\sin(y^2) = 2y\cos(y^2)$ (b) $\frac{d}{dx}\sin(x^2) = 2x\cos(x^2)$

(c) $\frac{d}{dx}\sin(y^2) = 2y\cos(y^2)$

3. On an exam, Jason was asked to differentiate the equation

$$x^2 + 2xy + y^3 = 7$$

Find the errors in Jason's answer: $2x + 2xy' + 3y^2 = 0$.

4. Which of (a) or (b) is equal to $\frac{d}{dx}(x \sin t)$?

(a) $(x\cos t)\frac{dt}{dx}$ (b) $(x\cos t)\frac{dt}{dx} + \sin t$

5. Determine which inverse trigonometric function g has the derivative

$$g'(x) = \frac{1}{x^2+1}$$

6. What does the following identity tell us about the derivatives of $\sin^{-1}x$ and $\cos^{-1}x$?

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

Exercises

1. Show that if you differentiate both sides of $x^2 + 2y^3 = 6$, the result is $2x + 6y^2\frac{dy}{dx} = 0$. Then solve for dy/dx and evaluate it at the point $(2, 1)$.

2. Show that if you differentiate both sides of $xy + 4x + 2y = 1$, the result is $(x+2)\frac{dy}{dx} + y + 4 = 0$. Then solve for dy/dx and evaluate it at the point $(1, -1)$.

In Exercises 3–8, differentiate the expression with respect to x, assuming that $y = f(x)$.

3. x^2y^3 4. $\frac{x^3}{y^2}$ 5. $(x^2+y^2)^{3/2}$

6. $\tan(xy)$ 7. $\frac{y}{y+1}$ 8. $e^{y/x}$

In Exercises 9–26, calculate the derivative with respect to x.

9. $3y^3 + x^2 = 5$ 10. $y^4 - 2y = 4x^3 + x$

11. $x^2y + 2x^3y = x + y$ 12. $xy^2 + x^2y^5 - x^3 = 3$

13. $x^3R^5 = 1$ 14. $x^4 + z^4 = 1$

15. $\frac{y}{x} + \frac{x}{y} = 2y$ 16. $\sqrt{x+s} = \frac{1}{x} + \frac{1}{s}$

17. $y^{-2/3} + x^{3/2} = 1$ 18. $x^{1/2} + y^{2/3} = -4y$

19. $y + \frac{1}{y} = x^2 + x$ 20. $\sin(xt) = t$

21. $\sin(x+y) = x + \cos y$ 22. $\tan(x^2y) = (x+y)^3$

23. $xe^y = 2xy + y^3$ 24. $e^{xy} = \sin(y^2)$

25. $\ln x + \ln y = x - y$ 26. $\ln(x^2+y^2) = x + 4$

In Exercises 27–30, compute the derivative at the point indicated without using a calculator.

27. $y = \sin^{-1}x, \quad x = \frac{3}{5}$ 28. $y = \tan^{-1}x, \quad x = \frac{1}{2}$

29. $y = \sec^{-1}x, \quad x = 4$ 30. $y = \arccos(4x), \quad x = \frac{1}{5}$

In Exercises 31–44, find the derivative.

31. $y = \sin^{-1}(7x)$ 32. $y = \arctan\left(\frac{x}{3}\right)$

33. $y = \cos^{-1}(x^2)$ 34. $y = \sec^{-1}(t+1)$

35. $y = x\tan^{-1}x$ 36. $y = e^{\cos^{-1}x}$

37. $y = \arcsin(e^x)$ 38. $y = \csc^{-1}(x^{-1})$

39. $y = \sqrt{1-t^2} + \sin^{-1}t$ 40. $y = \tan^{-1}\left(\frac{1+t}{1-t}\right)$

41. $y = (\tan^{-1}x)^3$ 42. $y = \frac{\cos^{-1}x}{\sin^{-1}x}$

43. $y = \cos^{-1}t^{-1} - \sec^{-1}t$ 44. $y = \cos^{-1}(x + \sin^{-1}x)$

45. Use Figure 8 to prove that $(\cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$.

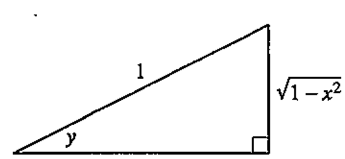

FIGURE 8 Right triangle with $y = \cos^{-1}x$.

46. Show that $(\tan^{-1} x)' = \cos^2(\tan^{-1} x)$ and then use Figure 9 to prove that $(\tan^{-1} x)' = (x^2 + 1)^{-1}$.

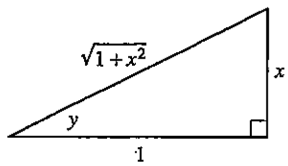

FIGURE 9 Right triangle with $y = \tan^{-1} x$.

47. Let $y = \sec^{-1} x$. Show that $\tan y = \sqrt{x^2 - 1}$ if $x \geq 1$ and that $\tan y = -\sqrt{x^2 - 1}$ if $x \leq -1$. *Hint:* $\tan y \geq 0$ on $\left(0, \frac{\pi}{2}\right)$ and $\tan y \leq 0$ on $\left(\frac{\pi}{2}, \pi\right)$.

48. Use Exercise 47 to verify the formula

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2 - 1}}$$

49. Show that $x + yx^{-1} = 1$ and $y = x - x^2$ define the same curve [except that $(0, 0)$ is not a solution of the first equation] and that implicit differentiation yields $y' = yx^{-1} - x$ and $y' = 1 - 2x$. Explain why these formulas produce the same values for the derivative.

50. Use the method of Example 4 to compute $\frac{dy}{dx}\big|_P$ at $P = (2, 1)$ on the curve $y^2x^3 + y^3x^4 - 10x + y = 5$.

In Exercises 51 and 52, find dy/dx at the given point.

51. $(x+2)^2 - 6(2y+3)^2 = 3, \quad (1, -1)$

52. $\sin^2(3y) = x + y, \quad \left(\frac{2-\pi}{4}, \frac{\pi}{4}\right)$

In Exercises 53–60, find an equation of the tangent line at the given point.

53. $xy + x^2y^2 = 6, \quad (2, 1)$

54. $x^{2/3} + y^{2/3} = 2, \quad (1, 1)$

55. $x^2 + \sin y = xy^2 + 1, \quad (1, 0)$

56. $\sin(x - y) = x\cos\left(y + \frac{\pi}{4}\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

57. $2x^{1/2} + 4y^{-1/2} = xy, \quad (1, 4)$

58. $x^2e^y + ye^x = 4, \quad (2, 0)$

59. $e^{2x-y} = \frac{x^2}{y}, \quad (2, 4)$

60. $y^2e^{x^2-16} - xy^{-1} = 2, \quad (4, 2)$

61. Find the points on the graph of $y^2 = x^3 - 3x + 1$ (Figure 10) where the tangent line is horizontal.

(a) First show that $2yy' = 3x^2 - 3$, where $y' = dy/dx$.

(b) Do not solve for y'. Rather, set $y' = 0$ and solve for x. This yields two values of x where the slope may be zero.

(c) Show that the positive value of x does not correspond to a point on the graph.

(d) The negative value corresponds to the two points on the graph where the tangent line is horizontal. Find their coordinates.

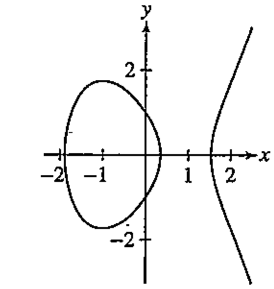

FIGURE 10 Graph of $y^2 = x^3 - 3x + 1$.

62. Show, by differentiating the equation, that if the tangent line at a point (x, y) on the curve $x^2y - 2x + 8y = 2$ is horizontal, then $xy = 1$. Then substitute $y = x^{-1}$ in $x^2y - 2x + 8y = 2$ to show that the tangent line is horizontal at the points $\left(2, \frac{1}{2}\right)$ and $\left(-4, -\frac{1}{4}\right)$.

63. Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal (Figure 11).

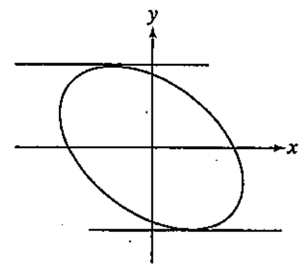

FIGURE 11 Graph of $3x^2 + 4y^2 + 3xy = 24$.

64. Show that no point on the graph of $x^2 - 3xy + y^2 = 1$ has a horizontal tangent line.

65. Figure 1 shows the graph of $y^4 + xy = x^3 - x + 2$. Find dy/dx at the two points on the graph with x-coordinate 0 and find an equation of the tangent line at $(1, 1)$.

66. **Folium of Descartes** The curve $x^3 + y^3 = 3xy$ (Figure 12) was first discussed in 1638 by the French philosopher-mathematician René Descartes, who called it the folium (meaning "leaf"). Descartes's scientific colleague Gilles de Roberval called it the jasmine flower. Both men believed incorrectly that the leaf shape in the first quadrant was repeated in each quadrant, giving the appearance of petals of a flower. Find an equation of the tangent line at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.

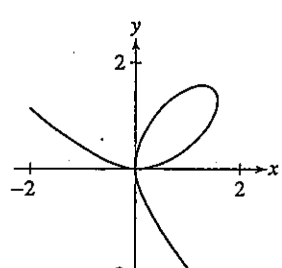

FIGURE 12 Folium of Descartes: $x^3 + y^3 = 3xy$.

67. Find a point on the folium $x^3 + y^3 = 3xy$ other than the origin at which the tangent line is horizontal.

68. GU Plot $x^3 + y^3 = 3xy + b$ for several values of b and describe how the graph changes as $b \to 0$. Then compute dy/dx at the point $(b^{1/3}, 0)$. How does this value change as $b \to \infty$? Do your plots confirm this conclusion?

69. Find the x-coordinates of the points where the tangent line is horizontal on the *trident curve* $xy = x^3 - 5x^2 + 2x - 1$, so named by Isaac Newton in his treatise on curves published in 1710 (Figure 13). *Hint:* $2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1)$.

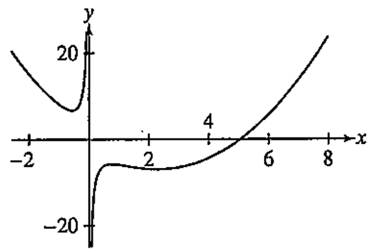

FIGURE 13 Trident curve: $xy = x^3 - 5x^2 + 2x - 1$.

70. Find an equation of the tangent line at each of the four points on the curve $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$ where $x = 1$. This curve (Figure 14) is an example of a *limaçon of Pascal*, named after the father of the French philosopher Blaise Pascal, who first described it in 1650.

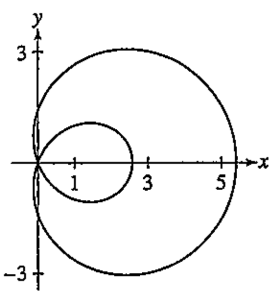

FIGURE 14 Limaçon: $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$.

71. Find the derivative at the points where $x = 1$ on the folium $(x^2 + y^2)^2 = \frac{25}{4}xy^2$. See Figure 15.

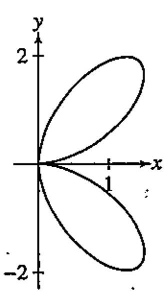

FIGURE 15 Folium curve: $(x^2 + y^2)^2 = \dfrac{25}{4}xy^2$.

72. CAS Plot $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ for $-4 \le x \le 4$, $-4 \le y \le 4$ using a computer algebra system. How many horizontal tangent lines does the curve appear to have? Find the points where these occur.

73. Calculate dx/dy for the equation $y^4 + 1 = y^2 + x^2$ and find the points on the graph where the tangent line is vertical.

74. Show that the tangent lines at $x = 1 \pm \sqrt{2}$ to the *conchoid* with equation $(x - 1)^2(x^2 + y^2) = 2x^2$ are vertical (Figure 16).

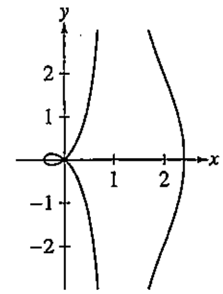

FIGURE 16 Conchoid: $(x - 1)^2(x^2 + y^2) = 2x^2$.

75. CAS Use a computer algebra system to plot $y^2 = x^3 - 4x$ for $-4 \le x \le 4$, $-4 \le y \le 4$. Show that if $dx/dy = 0$, then $y = 0$. Conclude that the tangent line is vertical at the points where the curve intersects the x-axis. Does your plot confirm this conclusion?

76. Show that for all points P on the graph in Figure 17, the segments $\overline{OP}$ and $\overline{PR}$ have equal length.

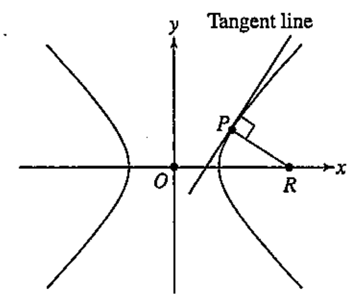

FIGURE 17 Graph of $x^2 - y^2 = a^2$.

In Exercises 77–80, use implicit differentiation to calculate higher derivatives.

77. Consider the equation $y^3 - \frac{3}{2}x^2 = 1$.

(a) Show that $y' = x/y^2$ and differentiate again to show that

$$y'' = \frac{y^2 - 2xyy'}{y^4}$$

(b) Express y'' in terms of x and y using part (a).

78. Use the method of the previous exercise to show that $y'' = -y^{-3}$ on the circle $x^2 + y^2 = 1$.

79. Calculate y'' at the point $(1, 1)$ on the curve $xy^2 + y - 2 = 0$ by the following steps:

(a) Find y' by implicit differentiation and calculate y' at the point $(1, 1)$.

(b) Differentiate the expression for y' found in (a). Then compute y'' at $(1, 1)$ by substituting $x = 1$, $y = 1$, and the value of y' found in (a).

80. Use the method of the previous exercise to compute y'' at the point $(1, 1)$ on the curve $x^3 + y^3 = 3x + y - 2$.

In Exercises 81–83, x and y are functions of a variable t and use implicit differentiation to relate dy/dt and dx/dt.

81. Differentiate $xy = 1$ with respect to t and derive the relation $\dfrac{dy}{dt} = -\dfrac{y}{x}\dfrac{dx}{dt}$.

82. Differentiate

$$x^3 + 3xy^2 = 1$$

with respect to t and express dy/dt in terms of dx/dt, as in Exercise 81.

83. Calculate dy/dt in terms of dx/dt.
(a) $x^3 - y^3 = 1$
(b) $y^4 + 2xy + x^2 = 0$

84. The volume V and pressure P of gas in a piston (which vary in time t) satisfy $PV^{3/2} = C$, where C is a constant. Prove that

$$\frac{dP/dt}{dV/dt} = -\frac{3}{2}\frac{P}{V}$$

The ratio of the derivatives is negative. Could you have predicted this from the relation $PV^{3/2} = C$?

Further Insights and Challenges

85. Show that if P lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$ (c, d constants), then the tangents to the curves at P are perpendicular.

86. The *lemniscate curve* $(x^2 + y^2)^2 = 4(x^2 - y^2)$ was discovered by Jacob Bernoulli in 1694, who noted that it is "shaped like a figure 8, or a knot, or the bow of a ribbon." Find the coordinates of the four points at which the tangent line is horizontal (Figure 18).

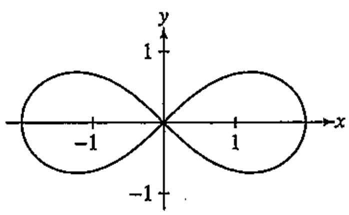

FIGURE 18 Lemniscate curve: $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

87. Divide the curve in Figure 19 into five branches, each of which is the graph of a function. Sketch the branches.

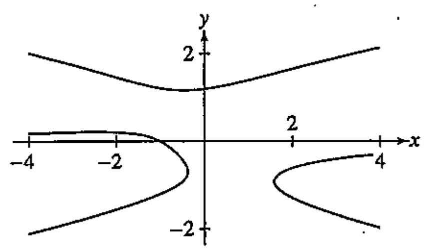

FIGURE 19 Graph of $y^5 - y = x^2y + x + 1$.

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