FINAL **TMath 124pm** Key Winter 2012

1. [6] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f be a function defined everywhere.

- T F If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = 0$, then $\lim_{x\to\infty}[f(x) - g(x)] = \infty$.
- T F If f is continuous at x, then f is differentiable at x.
- T F $\lim_{x\to 1} \frac{\log_2(x)}{x-1} = \lim_{x\to 1} \frac{(x-1)(\log_2(x))' - (\log_2(x))(x-1)'}{(x-1)^2}$ by L'Hospital's Rule.
- T F All local extrema numbers are also critical numbers.
- T F If f has a local minimum or maximum when $x = 4$, then $f'(4) = 0$.
- T F If f is such that $f'(4)$ DNE, then there is a local minimum or maximum when $x = 4$.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. [7] (Exam 2 #2) Sketch a graph and then *find a formula* of an example function f that satisfies the following conditions:

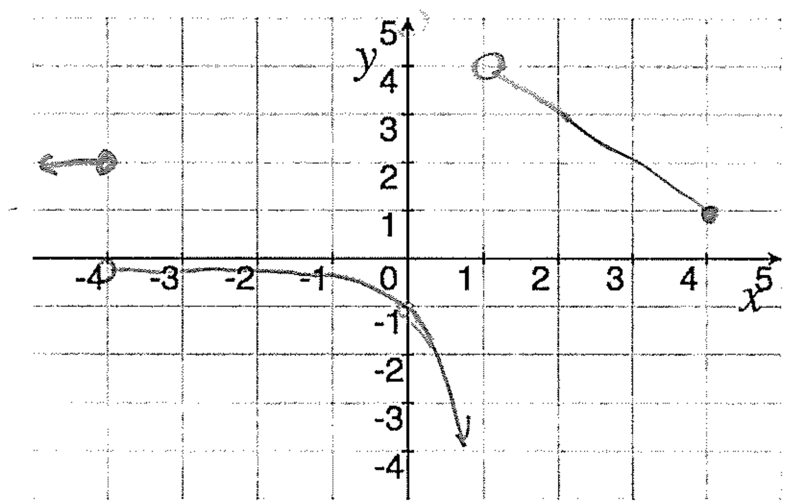
   (a) f is not differentiable when $x = -4$,

   (b) f is not continuous when $x = -4$,

   (c) $f'(3) = -1$, and

   (d) $\lim_{x\to 1^-} f(x) = -\infty$

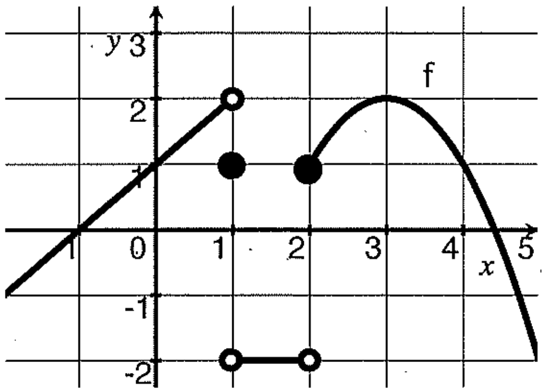
+4

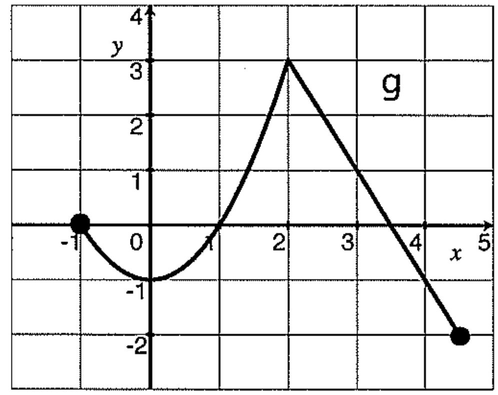


$$f(x) = \begin{cases} 2 & \text{if } x \le -4 \\ \frac{1}{x-1} & \text{if } -4 < x < 1 \\ -x+5 & \text{if } 1 < x \le 4 \end{cases}$$ +3

+.5 start.
+.5 form
+.5 a)
+.5 b)
+.5 c)
+.5 d

3. (Exam 1 #3) The graphs of f and g are shown below. Find the exact value (if possible):





$$f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1 & \text{if } x = 1 \\ -2 & \text{if } 1 < x < 2 \\ -(x-3)^2 + 2 & \text{if } 2 \le x \end{cases}$$

$$g(x) = \begin{cases} \overset{x^2-1}{\cancel{(x-1)^2}} & \text{if } -1 \le x < 2 \\ -2x+7 & \text{if } 2 \le x \le 4.5 \end{cases}$$

[1] (WebHW2#1)
$\lim_{x\to 1^+} f(x)$

-2

[1] (WebHW2#1)
$\lim_{x\to 2} g(x)$

3

[2] (§2.3 #2f)
$\lim_{x\to 3} \log_3(7 + f(x))$

$= \log_3(7 + \lim_{x\to 3} f(x))$ (+1)

$= \log_3(7+2) = \log_3 9 = 2$ (+1)

[3] (Derivative Wks)
$(f \cdot g)'(4)$ Product (+1)

$f'(4)g(4) + f(4)g'(4)$

$(-2)(-1) + (1)(-2)$
(+.5) (+.5) (+.5) (+.5)

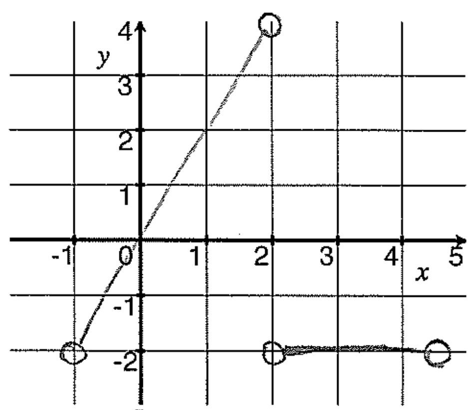
$2 - 2 = 0$

[3] (§3.4 #65)
$(f \circ g)'(4)$ Chain rule (+1)
notation (+.5)

$f'(g(4))\, g'(4)$

$f'(-1)(-2) = 0(-2) = 0$
(+.5) (+.5) (+.5)

[3] (PracticeFinal #4) Sketch the graph of $g'(x)$ on the blank set of axes to the right.

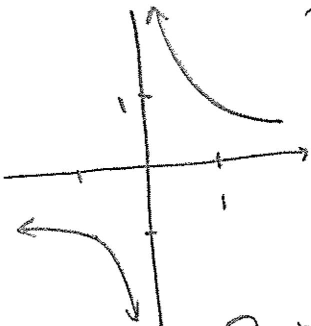


Shape (+.5) (+.5)
zero's (+.5)
got it (+.5) (+.5)
end points (.5)

4. Find the following *limits* if they exist. Make sure you show your work and justify your conclusions!

[3] (§2.2 Example 8)

$$\lim_{x\to 0^-} \frac{1}{x}$$



Recall the graph of $1/x$

so

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

or

By the big-little principal

notation (+.5)
justify (+1.5)
got it (+1)

[4] Quiz 3 #1

$$\lim_{x\to 0} \frac{\sin(6x)}{\sin(2x)\cos(6x)}$$

notation (+.5)
algebra (+1)
$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} =$ (+.5)

$$= \lim_{x\to 0} \frac{\sin 6x}{6x} \cdot \frac{6x}{2x} \cdot \frac{2x}{\sin(2x)} \cdot \frac{1}{\cos(6x)}$$ (+.5) (+.5) (+.5)

$$= \lim_{2x\to 0} \frac{\sin 6x}{6x} \cdot \lim_{x\to 0} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0} \frac{6}{2\cos(6x)} = 3$$

1 (+.5) 1 (+.5)

or

$$\overset{L'H}{=} \lim_{x\to 0} \frac{6\cos(6x)}{-\sin(2x)\,6\sin(6x) + 2\cos(2x)\cos(6x)}$$ (+1) (+1.5)

$$= \frac{6}{0+2} = 3$$

notation (+.5)
algebra (+1)

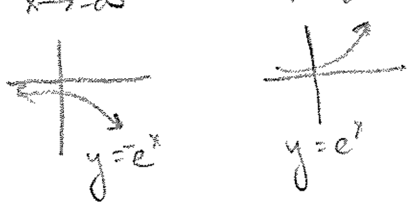
[4] (PracticeExam1 #4)

$$\lim_{x\to -\infty} e^x \sin x$$ (+.5)

Note $-1 \le \sin x \le 1$ for all x

$\Rightarrow$ $-e^x \le e^x \sin x \le e^x$ for all x (+.5)
(since e^x is always positive

Notice $\lim_{x\to -\infty} -e^x = 0 = \lim_{x\to -\infty} e^x$



Thus by the squeeze theorem (+1)

$$\lim_{x\to -\infty} e^x \sin(x) = 0$$ (+.5)

notation (+.5) sense (+1)

[3] (Limit Wks)

$$\lim_{x\to -2} \frac{2x^2+4x}{x+2}$$

notation (+.5)
alg (+1)
cancel (+.5)

$$= \lim_{x\to -2} \frac{2x\cancel{(x+2)}}{\cancel{x+2}}$$

got it (+1)

$$= \lim_{x\to -2} 2x = -4$$

or

$$\overset{L'H}{=} \lim_{x\to -2} \frac{4x+4}{1}$$

L'H (+1.5)
notation (+.5)

$$= 4(-2)+4 = -4$$

got it (+1)

5. Find $\frac{dy}{dx}$ for each of the following: (Do *not* simplify!)

[4] (§3.6 #47)
$y = (\sin x)^x$

into logs (+.5)
prop of logs (+.5)

$\ln y = x \ln(\sin x)$ product (+.5)

$\underbrace{\frac{1}{y} y'}_{(+.5)} = \underbrace{x \frac{\cos x}{\sin(x)}}_{\text{chain (+1)}} + \underbrace{\ln(\sin x)}_{(+.5)}$

$\Rightarrow y' = y\left[x\cot(x) + \ln(\sin x)\right]$

solved for y (+.5)

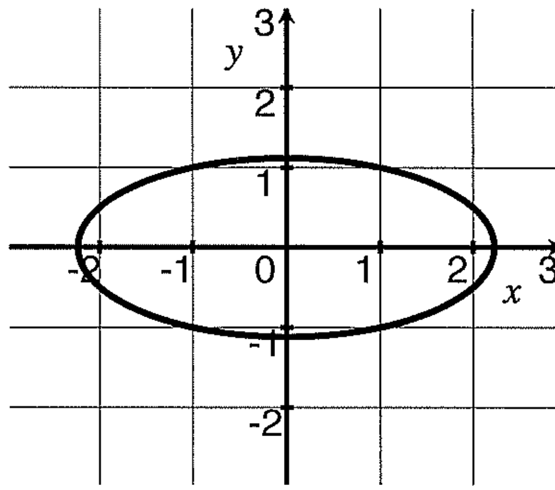
3 ~~[4]~~ (PracticeExam2 #8a)
$x^2 + 4y^2 = 5$

$2x + 8y\,y' = 0$
(+.5) (+1) (+.5)

$8yy' = -2x$

$y' = \frac{-2x}{8y} = \frac{-x}{4y}$

alg/solved for y' (+1)



(*)
$\left(-\frac{5}{2}\right)^2 + 4y^2 = 5$
$4y^2 = 5 - \frac{5}{2}$
$y^2 = \frac{5}{2} \cdot \frac{1}{4}$
$y = \pm\sqrt{5/8}$

6. The equation $x^2 + 4y^2 = 5$
defines an ellipse shown to the right.

(a) [3] (Exam 2 #7) Find the equation of the line tangent to the ellipse when $x = -1$ and $y < 0$.

Looking for $y = mx + b$ (+.5)

$m = y'\big|_{(-1,y)} = \frac{-(-1)}{4y}$ (+.5)

need to find y when $x = -1$

$\Rightarrow (-1)^2 + 4y^2 = 5$ (+.5)

$\Rightarrow y = \pm\sqrt{(5-1)/4} \Rightarrow y = \pm 1$ b/c $y < 0 \Rightarrow y = -1$ (+.5)

$= \frac{+1}{4(-1)} = \frac{-1}{4}$ (+.5)

So $y - (-1) = -1/4(x - (-1))$ or $y = -1/4\,x - 5/4$

plug in pl (+.5)

(b) ~~[3]~~ 4 (Derivative Wks #4) Find the points on the ellipse whose tangent lines are parallel to the line $2y + x = 4$.

alg to (+1.5) got them (+1)

$2y + x = 4$
$2y = -x + 4$
$y = -\frac{1}{2}x + 2$

find when $y' = -\frac{1}{2}$ (+.5)

so $\frac{-x}{4y} = -\frac{1}{2}$ (+.5)

or

$\frac{-x}{4\sqrt{\frac{5-x^2}{4}}} = -\frac{1}{2}$ (+.5)

$\frac{-x}{2\sqrt{5-x^2}} = -\frac{1}{2}$

$\frac{x}{\sqrt{5-x^2}} = 1$

$x = \sqrt{5-x^2}$

$x^2 = 5 - x^2$

$2x^2 = 5$

$x^2 = 5/2$

$x = \pm\sqrt{5/2}$

b/c $y > 0$ looking at graph when $x = -\sqrt{5/2}$ + $y = \sqrt{5/8}$ (*)

By symmetry also $(\sqrt{5/2}, -\sqrt{5/8})$

7. [3] (Quiz 5 #1) If $g(2) = 7$ and $-3 \leq g'(x) \leq 1$ for $2 \leq x \leq 5$, how small can $g(5)$ possibly be? Briefly justify your answer.

(+.5) $g'(x)$ is diff between 2+5 thus
$g(x)$ is con't
By the MVT there is a c (+1) between 2+5 so that

$$g'(c) = \frac{g(5)-g(2)}{5-2}$$

since $-3 \leq g'(c) \leq 1$

$\Rightarrow -3 \leq \frac{g(5)-g(2)}{5-2} \leq 1$ (+1)

$\Rightarrow -9 \leq g(5) - 7 \leq 3$ alg (+.5)

$\Rightarrow -2 \leq g(5) \leq 10$

8. Find the most general antiderivative for:

[2] (WebHW16 #1)
$y = x - 8$

$\frac{1}{2}x^2 - 8x + c$

(+1) (+.5) (+.5)

Check: $(\frac{1}{2}x^2 - 8x + c)'$

$\frac{1}{2} \cdot 2x - 8$ ✓

[2] (Lecture 3/5)
$y = 5^x \ln(5)$

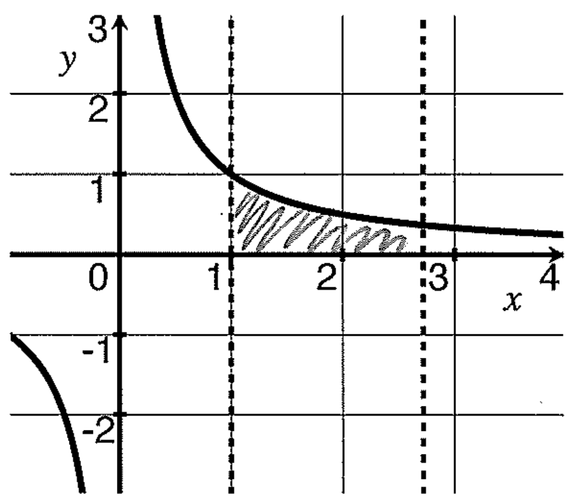
$5^x + c$

(+1) (+1)

Check: $(5^x + c)'$

$5^x \ln 5$ ✓

9. The graph of $y = \frac{1}{x}$ is shown to the right along with the vertical lines $x = e$ and $x = 1$.



(a) [3] (Lecture 3/5) Find $\int_1^e \frac{1}{x}\, dx$,

$$\int_1^e \frac{1}{x}\, dx = \ln x \Big]_1^e$$

$= \ln e - \ln 1$

$= 1 - 0 = 1$

FTC (+1)
anti der (+.5)
eval (+1)
notation (+.5)

(b) [1] (Lecture 3/5) Explain what you found in part (a) in terms of area.

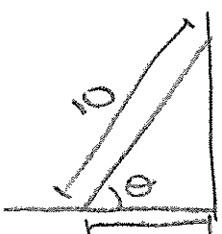
The shaded area is 1 unit^2 (+.5)

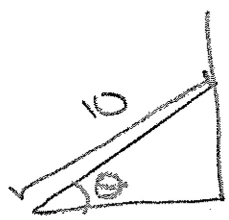
right area (+.5)

Note: the 1st problem is written up on the final from Winter 12 AM #10

10. [5] Choose only *ONE* of the following. Clearly identify which of the two you are answering and what work you want considered for credit.

- (Word Wks2 #10) A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of $12\text{ft}^3/\text{min}$, how fast is the water level rising when the water is 6 inches deep?
- (Exam2 #8) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the angle between the lader and the ground changing when the bottom of the ladder is 6ft from the wall?





Want $\left.\frac{d\theta}{dt}\right|_{x=6}$

$\leftarrow dx/dt = 1$ ft/s } interpret (+.5)

picture (+.5)

variables (+.5)

Sohcahtoa

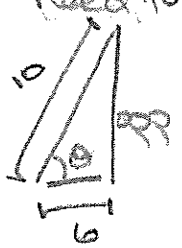
$\cos\theta = \frac{x}{10}$ (+.5)

$\frac{d}{dt}(\cos\theta) = \frac{d}{dt}\left(\frac{1}{10}x\right)$ wrt t (+.5)

$\underbrace{(-\sin\theta)}_{+.5}\frac{d\theta}{dt} = \underbrace{\frac{1}{10}\frac{dx}{dt}}_{+.5}$

$\Rightarrow \frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}\cdot\frac{-1}{\sin\theta}$

need to find θ when $x=6$ (+.5)



$\Rightarrow \cos\theta = \frac{6}{10}$

$\Rightarrow \theta = \arccos(3/5)$

note

$\sin\theta = \frac{8}{10} = \frac{\sqrt{10^2-36}}{10} = \frac{8}{10}$ found (+.5)

$\left.\frac{d\theta}{dt}\right|_{x=6} = \frac{1}{10}\cdot 1\cdot\frac{-1}{\sin(\arccos 3/5)}$

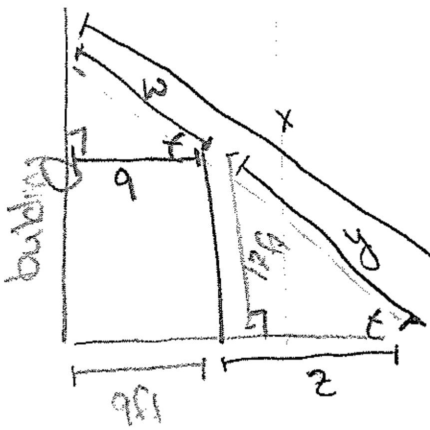
$= \frac{1}{\cancel{10}}\cdot\frac{-\cancel{10}}{8}$

$= -1/8$

plug in (+.5)

11. [5] Choose only *ONE* of the following. Clearly identify which of the two you are answering and what work you want considered for credit.

- A breeder has been selling 100 labradoodles a year at \$1500 each. A market survey indicated that for each increase in price by \$100, the number of labradoodles sold will decrease by 5 a year. Use calculus to find out what price the breeder should set so as to maximize his/her revenue?
- (Word Wks #1) A fence 17 ft tall runs parallel to the tall building at a distance of 9 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



want to minimize x

(+.5)

$$\frac{x}{9+z} = \frac{y}{z}$$

note $z^2+17^2=y^2$

$\Rightarrow z=\underbrace{\sqrt{y^2-17^2}}_{(+1)}$

sub (+.5)

$$\frac{x}{9+\sqrt{y^2-17^2}} = \frac{y}{\sqrt{y^2-17^2}}$$

or $x\sqrt{y^2-17^2} = 9y + y\sqrt{y^2-17^2}$

variables (+.5)
picture/started (+.5)

$$\Rightarrow x = \frac{y\left(9+\sqrt{y^2-17^2}\right)}{\sqrt{y^2-17^2}} = \frac{9y}{\sqrt{y^2-17^2}} + y$$

$$dx/dy = 9y\,\tfrac{-1}{2}(y^2-17^2)^{-3/2}\cdot 2y + 9(y^2-17^2)^{-1/2} + 1$$

$$dx/dy = \frac{-9\cdot y^2}{(y^2-17^2)^{3/2}} + \frac{9}{(y^2-17^2)^{1/2}} + 1$$

$$dx/dy = \frac{1}{(y^2-17^2)^{3/2}}\left[\underbrace{-9y^2+9(y^2-17^2)+(y^2-17^2)^{3/2}}\right]$$

CP (+1)
justify max (+1)

CP: DNE when $y=17$
and when $dy/dx=0$

$0 = \cancel{-9y^2} + \cancel{9y^2} - 9\cdot17^2 + (y^2-17^2)^{3/2}$

$[9\cdot17^2]^{2/3} = [(y^2-17^2)^{3/2}]^{2/3}$

$51\sqrt[3]{51} = y^2-17^2$

$\Rightarrow y^2 = 51\sqrt[3]{51}+17^2$

Notice

17   25.6

$dx/dy\big|_{y=20}$ $-$ $dx/dy\big|_{y=30}$ $+$

length is minimal when $y=25.6$ ft

$\Rightarrow z=19.14$ and $x=37.7$ ft