Exam 1 — TMath 124 — Winter 2012 — Key

1. [4] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be functions and x and y be positive numbers.

   T **(F)** $\dfrac{2x}{6x+y} = \dfrac{x}{3x+y}$ $\qquad \dfrac{x}{3x+y} = \dfrac{2x}{2(3x+y)} = \dfrac{2x}{6x+2y}$

   T **(F)** $\dfrac{x^2+x-6}{x-2} = x+3$ $\qquad$ notice on the left 2 can't be plugged in for x but on the right if x=2, we have 5

   (T) F $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$ where a is in the domain of f and g.

   (T) F If $f'(a)$ exists, then $\underbrace{\lim\limits_{x\to a} f(x) = f(a)}_{\text{continuity}}$.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [6] (§2.2 #16, §2.5 #8, & §2.7 #21) Sketch the graph of an example function f that satisfies the following conditions:

   (+1) (a) f is continuous everywhere but when $x = 2$ and $x = 4$.

   (+1) (b) $f(2) = 4$

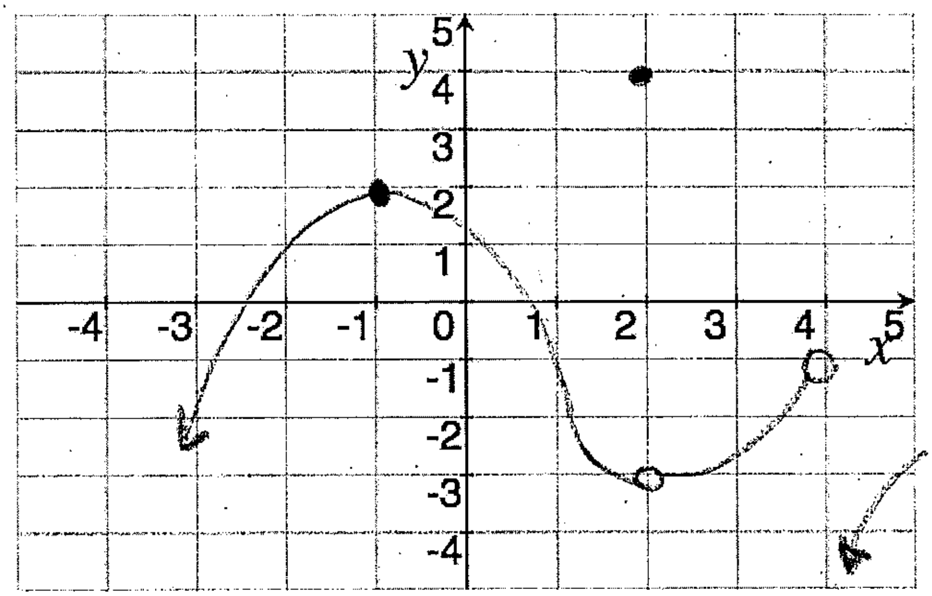
   (+1) (c) $\lim\limits_{x\to 2} f(x) = -3$

   (+1) (d) $\lim\limits_{x\to 4^+} f(x) = -\infty$

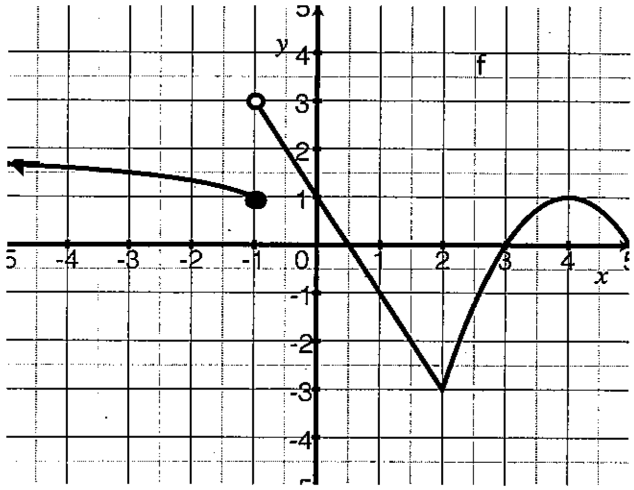
   (+1) (e) $f(-1) = 2$

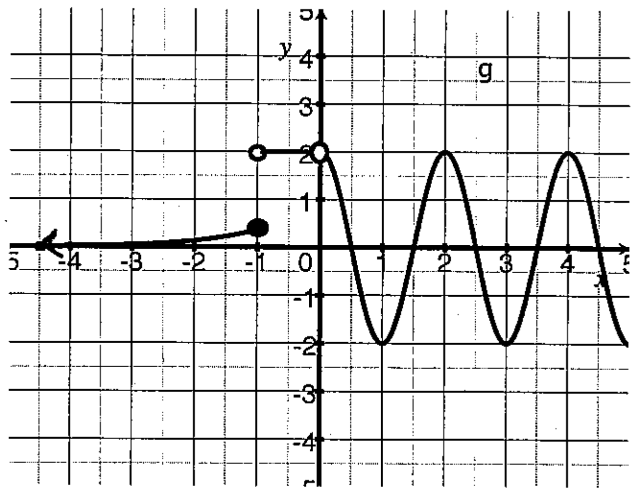
   (+1) (f) $f'(-1) = 0$

one answer...



3. [14] The graphs of f and g and their rules are provided below:





$$f(x) = \begin{cases} \frac{2x}{x-1} & \text{if } x \le \cancel{x} -1 \\ -2x+1 & \text{if } -1 < x \le 2 \\ -(x-4)^2+1 & \text{if } 2 < x \end{cases} \qquad g(x) = \begin{cases} e^x & \text{if } x \le -1 \\ 2 & \text{if } -1 < x < 0 \\ 2\cos(\pi x) & \text{if } 0 < x \end{cases}$$

Use the graphs above to ~~find~~ Estimate the following (if they exist!):

[2] (WebHW2 #3) $\lim_{x\to -1^-} g(x)$

$e^{-1} = 1/e \approx 1/2$ (+1)

Correct graph (.5)
notation (+.5)

[2] (WebHW5 #1) $\lim_{x\to -\infty} f(x)$

2 (+1) horiz asy (+.5)
correct graph (+.5)

[3] (§2.3 #2) $\lim_{x\to 0}[6g(x) - f(x)]$ (+.5) (+.5) (+.5)

limit law (+1)
notation (.5) $= 6\lim_{x\to 0} g(x) - \lim_{x\to 0} f(x) = 6\cdot 2 - 1 = 11$

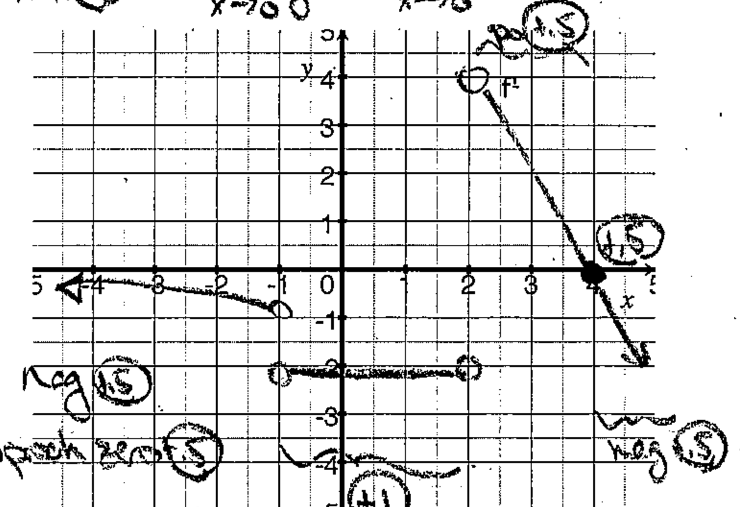
[3] (WebHW5 #9) $f'(1)$

(+1) slope of line tangent to f when $x=1$

so -2 (+.5) (+.5)

correct graph (.5)
notation (+.5)

[4] (§2.8 #9) Make a rough sketch of the graph of $f'(x)$:



endpoints (.5)

4. [12] Find the limit or explain why it does not exist.

[1] (WebHW4 #4) $\lim_{x\to 9} \frac{40+\sqrt{x}}{\sqrt{40+x}}$

$= \frac{40+\sqrt{9}}{\sqrt{40+9}}$ by property 5/cont.

$= \frac{40+3}{\sqrt{49}} = \frac{43}{7}$

(+1)

[3] (WebHW3 #7) $\lim_{x\to -5} \frac{\frac{1}{5}+\frac{1}{x}}{5+x}$

$= \lim_{x\to -5} \frac{\frac{x+5}{5x}}{5+x}$ alg frac (+.5)

$= \lim_{x\to -5} \frac{x+5}{5x} \div \frac{(5+x)}{1}$ alg ÷ (+.5)

$= \lim_{x\to -5} \frac{\cancel{x+5}}{5x} \cdot \frac{1}{\cancel{5+x}}$ alg simp (+.5)

$= \lim_{x\to -5} \frac{1}{5x} = \frac{1}{-25}$ limit law (+1)

notation (+.5)

[4] (InfLimits Wks) $\lim_{x\to -\infty} \frac{x^7+x}{x^5-4}$

$= \lim_{x\to -\infty} \frac{x^7+x}{x^5-4} \frac{(1/x^5)}{(1/x^5)}$ multiplier (+1)

$= \lim_{x\to -\infty} \frac{x^2 + 1/x^4}{1 - 4/x^5}$ alg (+.5)

limit laws (+1) $= \frac{\lim_{x\to -\infty} x^2 + \lim_{x\to -\infty} 1/x^4}{\lim_{x\to -\infty} 1 - \lim_{x\to -\infty} 4/x^5}$

$= \lim_{x\to -\infty} x^2 = \infty$ (+.5)

correct ∞ (+.5)

notation (+.5)

[4] (Practice #4) $\lim_{x\to\infty} e^{-2x}\sin(x)$

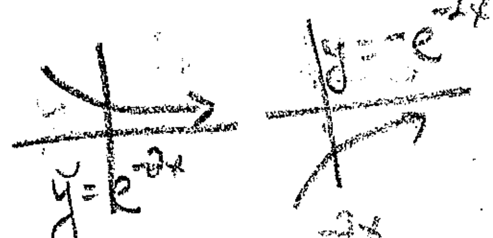
notation / sense (+.5)

(+.5) Recall $-1 \le \sin x \le 1$ (*)

(+.5) Since $e^{-2x} > 0$ for all x we can multiply the ineq (*) by e^{-2x} to get.

(+.5) $-e^{-2x} \le e^{-2x}\sin x \le e^{-2x}$,

Since



(+1) $\lim_{x\to\infty} e^{-2x} = \lim_{x\to\infty} -e^{-2x} = 0$

Thus by the Squeeze Thm (+.5)

$\lim e^{-2x}\sin x = 0.$ (+.5)

5. [3] (CalcWebHW5 #10) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 6$ if $g(6) = -2$ and $g'(6) = 3$.

(+.5) Equation of a tangent line: $y = mx + b$

(+1) m = slope of line tangent to g when $x=6$ $= g'(6) = 3$

So we know $m = 3$.

(+.5) Since the tangent line passes thru $(6, g(6)) = (6, -2)$ we know

$-2 = 3(6) + b$ (+.5)

$\Rightarrow b = 20$

So $y = 3x - 20$ (+.5)

or

$y + 2 = 3(x-6)$

6. Let $f(x) = x^2 - x$.

(a) [2] (§3.1 #17) Find $f'(x)$

$f'(x) = (x^2 - x)' = (x^2)' - (x)' = 2x - 1$ (+1) (+1)

by the power rule

(b) [4] (§2.8 #29) Find the derivative of f using the definition of derivative. That is, use $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ and verify your answer to part (a).

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^2 - (x+h)] - [x^2 - x]}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-x} - h \cancel{-x^2} \cancel{+x}}{h}$$

.5 .5 .5 .5 (+2) alg (+1)

$$= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h-1)}{\cancel{h}} = \lim_{h\to 0} (2x + h - 1)$$

prop (+.5)

$= 2x + 0 - 1$

$= 2x - 1$ (+.5) match !! :)

7. [5] (Story Problem Worksheet) Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit. No, doing both questions will not earn you extra credit.

   (a) A tank contains 5000L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25L/min. Find a function that records the concentration of salt after t minutes (in grams per liter) and then find out what happens to the concentration as $t \to \infty$.

   (b) Explain how scientists know there are at least two points directly opposite each other on the surface of the earth that are the same temperature.

a) Let t be the time (in minutes) since the Brine started flowing.

(+1) Recall concentration $= \dfrac{\text{salt in grams}}{\text{liquid in Liters}}$.

Liquid in tank at time t

(+1) $=$ amount at beginning $+$ amount added

$= 5000L + 25 \tfrac{L}{min} \cdot t \text{ min}$

$= (5000 + 25t)\ L$

Salt in tank at time t

(+1) $=$ amount at beginning $+$ amount added

$= 0 + 30 \tfrac{g}{L} \cdot 25 \tfrac{L}{min} \cdot t \text{ min}$

$= (30 \cdot 25t)\ g$

using our computations from the right side $\longrightarrow$

(+.5) Concentration $= \dfrac{30 \cdot 25t}{5000 + 25t} = \boxed{\dfrac{30t}{200 + t}}$

To find $\lim\limits_{t \to \infty}$ concentration we compute

(+.5) $\lim\limits_{t \to \infty} \dfrac{30t}{200+t} \dfrac{(1/t)}{(1/t)} = \lim\limits_{t \to \infty} \dfrac{30}{\frac{200}{t} + 1}$ (+.5)

$= \dfrac{\lim\limits_{t \to \infty} 30}{\lim\limits_{t \to \infty} \left(\frac{200}{t} + 1\right)} = \boxed{\dfrac{30}{1}}$ (+.5)

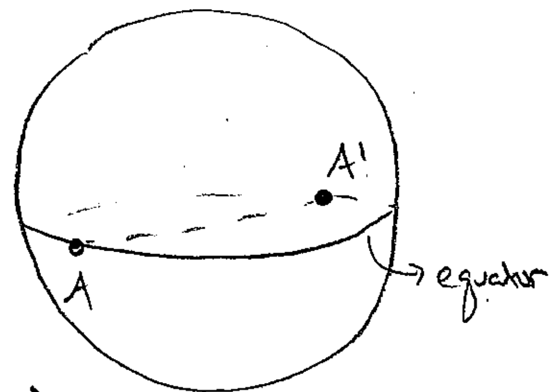
Thus the limit of concentration of salt as time goes to infinity is 30 g/L.

b) Let A be a location and A' a location directly opposite A on the earth. For simplicity let's confine ourselves to locations on the equator.

start (+.5)

Let Temp be the function that returns the temperature given a location.

define functions (+1)



Consider Temp(A) - Temp(A').

con't (+1)

Certainly Temp is continuous with respect to location. (The temperature never jumps instantaneously as we walk from place to place).

Since the difference of 2 continuous functions is continuous we know

sense (+1.5)

$F(A) = \text{Temp}(A) - \text{Tem}(A')$ is continuous.

Consider moving around the equator.

(+1) IVT

If $F(A) = 0$ then $\text{Tem}(A) = \text{Tem}(A')$ & we've located 2 locations on opposite sides of the earth that are the same temp.

If $F(A) \neq 0$, without loss of generality assume $F(A) > 0$. Then if we travel along the equator, we will eventually reach A'.

Since $F(A) = \text{Temp}(A) - \text{Tem}(A') > 0$

$\Rightarrow F(A') = \text{Temp}(A') - \text{Temp}(A) < 0$

The IVT implies there exists some location C so that $F(C) = 0$.