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8:37
44 min

FINAL **TMath 124am** Key Winter 2012

1. [6] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f be a function defined everywhere.

ex $f(x)=x^2$ and $g(x)=x$

T **F** If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}[f(x) - g(x)] = 0$.

T F If f is differentiable at x, then f is continuous at x.

$\lim_{x\to 1} \frac{\log_2(x)}{x-1} \overset{L'H}{=} \lim_{x\to 1} \frac{(\log_2(x))'}{(x-1)'}$

T **F** $\lim_{x\to 1} \frac{\log_2(x)}{x-1} = \lim_{x\to 1} \frac{(x-1)(\log_2(x))' - (\log_2(x))(x-1)'}{(x-1)^2}$ by L'Hospital's Rule.

T F All local extrema numbers are also critical numbers.

T **F** If f has a local minimum or maximum when $x = 4$, then $f'(4) = 0$.

T **F** If f is such that $f'(4) = 0$, then there is a local minimum or maximum when $x = 4$.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

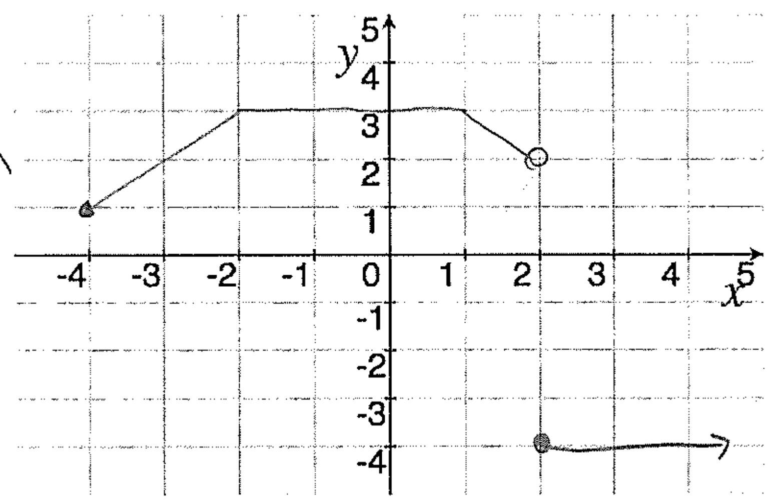
2. [7] (Exam 2 #2) Sketch a graph and then *find a formula* of an example function f that satisfies the following conditions:

(a) f is not differentiable when $x = 1$, ✓ +1

(b) f is continuous when $x = 1$, ✓ +1

(c) $f'(-3) = 1$, and ✓ +1

(d) $\lim_{x\to\infty} f(x) = -4$ ✓ +1

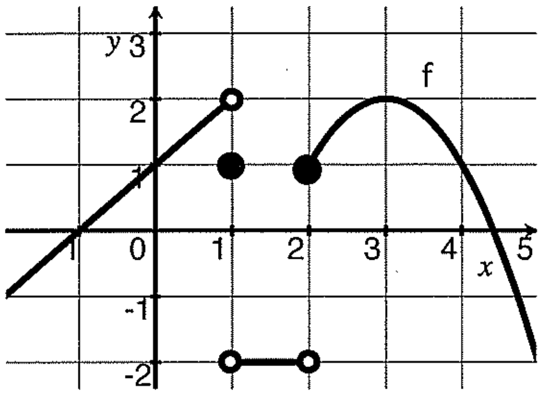


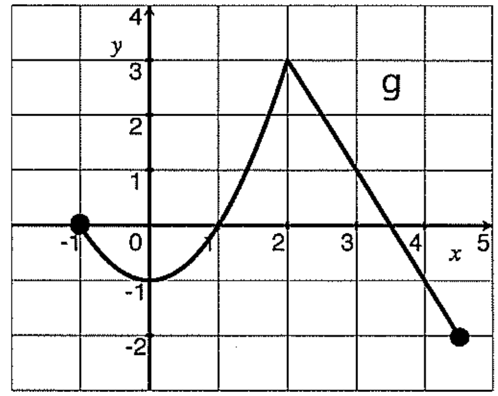
$$f(x) = \begin{cases} x+5 & \text{if } -4 \le x \le -2 \\ 3 & \text{if } -2 < x < 1 \\ -x+4 & \text{if } 1 \le x < 2 \\ -4 & \text{if } 2 \le x \end{cases}$$

+3

+.5 form
+.5 start
+.5 a)
+.5 b)
+.5 c)
+.5 d)

3. (Exam 1 #3) The graphs of f and g are shown below. Find the exact value (if possible):





$$f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1 & \text{if } x = 1 \\ -2 & \text{if } 1 < x < 2 \\ -(x-3)^2 + 2 & \text{if } 2 \le x \end{cases}$$

$$g(x) = \begin{cases} \overset{x^2-1}{\cancel{(x-1)^2}} & \text{if } -1 \le x < 2 \\ -2x+7 & \text{if } 2 \le x \le 4.5 \end{cases}$$

[1] (WebHW2#1)
$\lim_{x \to 1^+} f(x)$

-2

[1] (WebHW2#1)
$\lim_{x \to 2} g(x)$

3

[2] (§2.3 #2f)
$\lim_{x \to 3} \sqrt{7 + f(x)}$

$= \sqrt{7 + \lim_{x \to 3} f(x)}$ (+1)

$= \sqrt{7+2} = \sqrt{9} = 3$ (+1)

[3] (Derivative Wks)
$(f \cdot g)'(4)$ product rule (+1)

$f'(4)g(4) + f(4)g'(4)$

$(-2)(-1) + (1)(-2)$
(+.5) (+.5) (+.5) (+.5)

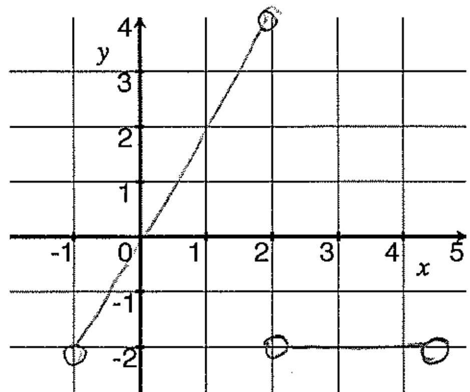
$2 - 2 = 0$

[3] (§3.4 #65)
$(f \circ g)'(4)$ Chain rule (+1)
notation (+.5)

$f'(g(4))g'(4)$

$f'(-1)\,(-2) = \phi(-2) = 2$
(+.5) (+.5) (+.5)

[3] (PracticeFinal #4) Sketch the graph of $g'(x)$ on the blank set of axes to the right.



shape (+.5) (+.5)
zero's (+.5)
pt A (+.5) (+.5)
end pts (+.5)

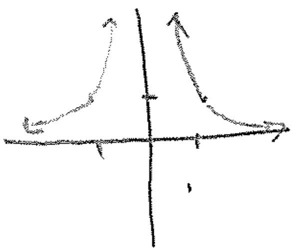
notation (+.5)
algebra (+1)
$\lim_{\theta\to0}\frac{\sin\theta}{\theta} =$ (+.5)

4. Find the following *limits* if they exist. Make sure you show your work and justify your conclusions!

[3] (§2.2 Example 8)

$$\lim_{x\to0}\frac{1}{x^2}$$

Recall the graph of $1/x^2$:



so $\lim_{x\to0} 1/x^2 = \infty$

or

By the big-little principal

notation (+.5)
justify (+1.5)
got it (+1)

[4] Quiz 3 #1

$$\lim_{x\to0}\frac{\sin(3x)\sin(5x)}{x^2}$$ (+.5) (+.5)

$$= \lim_{x\to0} \frac{\sin(3x)}{3x}\,\frac{3x\cdot5x}{x^2}\,\frac{\sin 5x}{5x}$$

$$= \lim_{x\to0}\frac{\sin 3x}{3x}\cdot\lim_{x\to0}\frac{15x^2}{x^2}\cdot\lim_{x\to0}\frac{\sin 5x}{5x}$$

1 (+.5) 1 (+.5)

$= 15$

or

$$\overset{L'H}{=} \lim_{x\to0}\frac{5\sin(3x)\cos(5x)+3\cos(3x)\sin(5x)}{2x}$$ (+1) "0/0" (+.5)

$$\overset{L'H}{=} \lim_{x\to0}\frac{-5\sin(3x)\cdot5\sin(5x)+5\cdot3\cos(3x)\cos5x+5\cdot3\cos(3x)\cos(5x)-3\cdot3\sin(3x)\sin(5x)}{2}$$ (+.5) (+.5) (.5)

$$= \frac{0+15+15-0}{2} = 15$$ notation (+.5)

[4] (PracticeExam1 #4)

$$\lim_{x\to-\infty} e^x\sin x$$

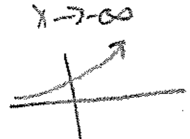
Note $-1 \le \sin x \le 1$ for all x (+.5)

$\Rightarrow$ (+.5) $-e^x \le e^x\sin x \le e^x$ for all x

(since e^x is always positive).

Notice $\lim_{x\to-\infty} -e^x = 0 = \lim_{x\to-\infty} e^x$

$-e^x = y$



$e^x = y$

Thus by the squeeze theorem (+1)

$\lim_{x\to-\infty} e^x\sin(x) = 0$ (+.5)

notation (+.5) sense (+1)

[3] (Quiz 1 #2)

notation (+.5)
alg (+1)

$$\lim_{x\to1}\frac{x^3-1}{x^2-1}$$

$$= \lim_{x\to1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$$ (+.5) (+.5)

$$= \lim_{x\to1}\frac{x^2+x+1}{x+1} = 3/2$$ got it (+.5)

or

$$\overset{L'H}{=} \lim_{x\to1}\frac{3x^2}{2x}$$ L'H (+1.5) notation (+.5) got it (+1)

$$= \lim_{x\to1}\frac{3x}{2}$$

$= 3/2$

5. Find $\frac{dy}{dx}$ for each of the following: (Do *not* simplify!)

intro logs (+.5)
Prop logs (+.5)

[4] (§3.6 #47)
$y = (\cos x)^x$

$\ln y = x \ln(\cos x)$ product (+.5)

$\underbrace{\frac{1}{y} \cdot y'}_{(+.5)} = x \underbrace{\frac{-\sin x}{\cos x}}_{\text{chain } (+1)} + \underbrace{\ln(\cos(x))}_{(+.5)}$

$\Rightarrow y' = y\left[-x\tan(x) + \ln\cos(x)\right]$

solved for y' (+.5)

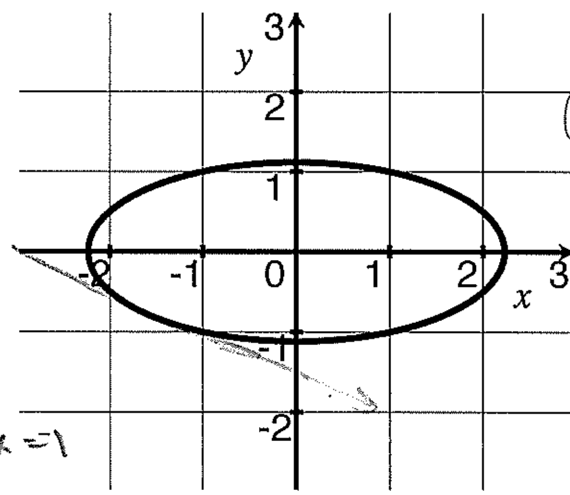
3 ~~[4]~~ (PracticeExam2 #8a)
$x^2 + 4y^2 = 5$

$\underset{(+.5)}{2x} + \underset{(+1)}{4\,2y\,y'} = \underset{(+.5)}{0}$

$8yy' = -2x$

$y' = \frac{-2x}{8y} = \frac{-x}{4y}$

alg / solved for y' (+1)



(★)
$\left(\sqrt{\frac{5}{2}}\right)^2 + 4y^2 = 5$
$4y^2 = 5 - 5/2$
$y^2 = 5/2 \cdot 1/4$
$y = \pm\sqrt{5/8}$

6. The equation $x^2 + 4y^2 = 5$ defines an ellipse shown to the right.

(a) [3] (Exam 2 #7) Find the equation of the line tangent to the ellipse when $x = -1$ and $y < 0$.

Looking for $y = mx + b$ (+.5)

$m = y'\big|_{(-1,y)} = \frac{-(-1)}{4y}$ (+.5)

need to find y when $x = -1$

$\Rightarrow (-1)^2 + 4y^2 = 5$

$\Rightarrow y = \pm\sqrt{(5-1)/4} \Rightarrow y = \pm 1$ (+.5) b/c $y < 0 \Rightarrow y = -1$ (+.5)

$= \frac{+1}{4(-1)} = -1/4$ (+.5)

So $y - (-1) = -1/4\,(x - (-1))$ or $y = -1/4\,x - 5/4$ plug in pt (+.5)

(b) 4 ~~[3]~~ (Derivative Wks #4) Find the points on the ellipse whose ~~tangent lines~~ are parallel to the line $2y + x = 4$.

$2y + x = 4$
$2y = -x + 4$
$y = -1/2\, x + 2$

find when $y' = -1/2$ (+.5)

So $\frac{-x}{4y} = -1/2$ (+.5)

or $\frac{-x}{4\sqrt{\frac{5-x^2}{4}}} = -1/2$ (+.4)

$\frac{-x}{2\sqrt{5-x^2}} = -1/2$ algebra (+1.5)

$\frac{x}{\sqrt{5-x^2}} = 1$

$x = \sqrt{5-x^2}$

$x^2 = 5 - x^2$

$2x^2 = 5$

$x^2 = 5/2$

got them (+1)
$x = \pm\sqrt{5/2}$

b/c $y > 0$
looking at graph
when $x = +\sqrt{5/2}$ (★)
& $y = \sqrt{5/8}$

By Symmetry
also at $\left(\sqrt{5/2}, -\sqrt{5/8}\right)$

7. [3] (WebHW13 #12) If $g(2) = 7$ and $-3 \le g'(x) \le 1$ for $2 \le x \le 5$, how small can $g(5)$ possibly be? Briefly justify your answer.

+.5 $g'(x)$ is diff between 2 & 5 thus $g(x)$ is con't
By the MVT +1 there is a c between 2 & 5 so that

$$g'(c) = \frac{g(5)-g(2)}{5-2}$$

Since $-3 \le g'(c) \le 1$

$\Rightarrow -3 \le \frac{g(5)-g(2)}{5-2} \le 1$ +.1

$\Rightarrow -9 \le g(5) - 7 \le 3$ alg +.5

$\Rightarrow -2 \le g(5) \le 10$

8. Find the most general antiderivative for:

[2] (WebHW16 #1)
$y = x - 8$

$\frac{1}{2}x^2 - 8x + C$ +1 +.5 +.5

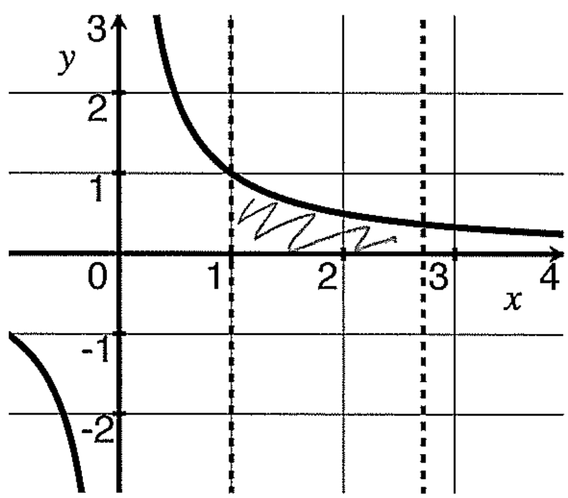
Check: $(\frac{1}{2}x^2 - 8x + c)'$
$\frac{1}{2}2x - 8$ ✓

[2] (Lecture 3/5)
$y = 5^x \ln(5)$

$5^x + c$ +1 +.1

check $(5^x + c)'$
$5^x \ln 5$ ✓

9. The graph of $y = \frac{1}{x}$ is shown to the right along with the vertical lines $x = e$ and $x = 1$.



(a) [3] (Lecture 3/5) Find $\int_1^e \frac{1}{x}\,dx$,

$$\int_1^e \frac{1}{x}\,dx = \ln x \Big]_1^e$$

FTC +1
anti der +.5

$= \ln e - \ln 1$ eval +1

$= 1 - 0 = 1$ notation +.5

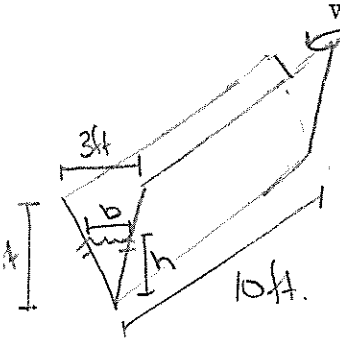
(b) [1] (Lecture 3/5) Explain what you found in part (a) in terms of area.

The shaded area is 1 unit^2 right area +.5

+.5

10. [5] Choose only *ONE* of the following. Clearly identify which of the two you are answering and what work you want considered for credit.

- (Word Wks2 #10) A trough is 10 ft long and its ends have the shape of isosceles triangles that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of $12\text{ft}^3/\text{min}$, how fast is the water level rising when the water is 6 inches deep?
- (Exam2 #8) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the angle between the lader and the ground changing when the bottom of the ladder is 6ft from the wall?



$\frac{dV}{dt} = 12\,\text{ft}^3/\text{min}$ } interpret (+.5)

find $\left.\frac{dh}{dt}\right|_{h=.5\text{ft}}$ unit convert (+.5)

picture (+.5)
variables (+.5)

Volume of Water $V = \frac{1}{2} \cdot b \cdot h \cdot 10$ (+.5)

relation between b & h:

$\frac{b}{h} = \frac{3}{1}$ by similar Δ's (+.5)

$\Rightarrow b = 3h$

$V = \frac{1}{2}(3h)h \cdot 10$ sub (+.5)

$V = 15h^2$

$\Rightarrow \frac{dV}{dt} = 30h \frac{dh}{dt}$ der (+1) +.5 w.r.t t

So $12\,\text{ft}^3/\text{min} = 30 \cdot .5 \frac{dh}{dt}$

$\Rightarrow \frac{12}{30 \cdot .5} = \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{2}{25}\,\text{ft}/\text{sec}$

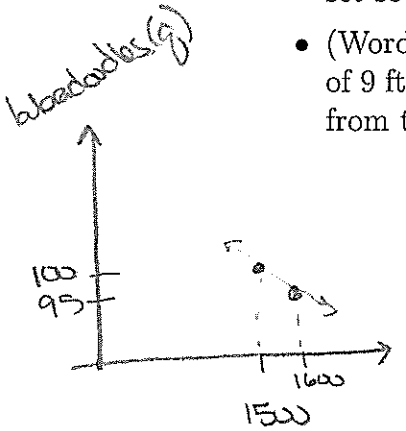
plug in correctly (.5)

note: the 2nd problem is written up on the final from Winter 12 PM

#10.

11. [5] Choose only *ONE* of the following. Clearly identify which of the two you are answering and what work you want considered for credit.

- A breeder has been selling 100 labradoodles a year at \$1500 each. A market survey indicated that for each increase in price by \$100, the number of labradoodles sold will decrease by 5 a year. Use calculus to find out what price the breeder should set so as to maximize his/her revenue?
- (Word Wks #1) A fence 17 ft tall runs parallel to the tall building at a distance of 9 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



Revenue = $p \cdot q$ (+.5)   want to maximize revenue

relation between p & q: line

$q - 100 = \frac{-5}{100}(p - 1500)$

$\Rightarrow q = \frac{-1}{20}p + 5 \cdot 15 + 100$

$\Rightarrow q = -\frac{1}{20}p + 175$ (+1)

$\begin{array}{r} 15 \\ 5 \\ \hline 75 \end{array}$

variables (+.5)
stated (+.5)

sub (+.5)

So $Rev = p\left(-\frac{1}{20}p + 175\right)$

$= -\frac{1}{20}p^2 + 175p$

$dRev/dp = -\frac{2}{20}p + 175$

Critical Points: (+1)

$0 = -\frac{2}{20}p + 175$

$p = 175 \cdot 10$

$= \$1750$

Justify (+1) is a maximum b/c Rev function is parabola opening down

revenue is

$1750 \cdot (175 - 84.5)$

$1750 \cdot \underbrace{90.5}$

note price will have to change b/c selling half a dog