

*Key*

Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let  $f$  and  $g$  be differentiable functions and  $h$  be a constant.

T  F  $\frac{x+h}{2x} = \frac{1+h}{x}$

T  F  $\sqrt{x^2 + h^2} = x + h$

T  F  $\lim_{x \rightarrow r} f(x) = f(r)$  for all  $r$  in the domain of  $f$ .

T  F If  $\lim_{x \rightarrow r} g(x) = 0$ , then  $\lim_{x \rightarrow r} \frac{f(x)}{g(x)}$  does not exist.

T  F  $\frac{d}{dx}(\frac{1}{x}) = -1$

$$\frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$$

Since  $f$  is differentiable  
on its also continuous

$$\lim_{x \rightarrow 0} \frac{x(x+3)}{x} = \lim_{x \rightarrow 0} (x+3) = 3$$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

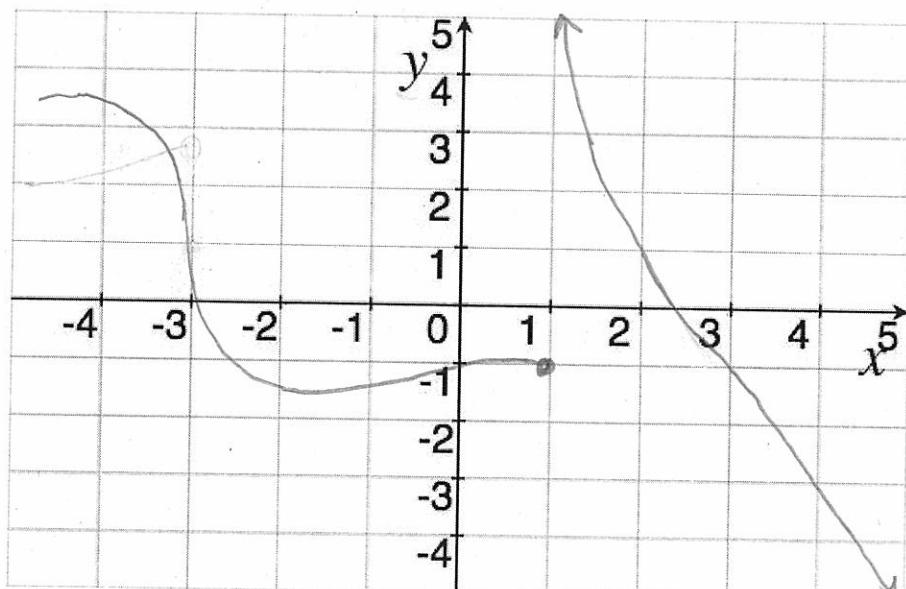
2. [5] Sketch the graph of an example function  $f$  that satisfies the following conditions:

(a)  $f$  is not differentiable when  $x = -3$

(b)  $f$  is continuous when  $x = -3$

(c)  $\lim_{x \rightarrow 1^+} f(x) = \infty$

(d)  $f'(3) = -2$



3. Compute the following limits:

$$(a) \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{2x^2 - 8x + 6} \quad [\text{den} = 0]$$

$$= \lim_{x \rightarrow 1} \frac{(x+2)(x-1)}{2(x-3)(x-1)}$$

$$= \lim_{x \rightarrow 1} \frac{x+2}{2(x-3)} = \frac{1+2}{2(1-3)} = \frac{3}{-4}$$

$$= \frac{3}{-4}$$

$$(c) \lim_{\theta \rightarrow 0^+} \frac{\theta + \theta^2}{1 - \cos \theta} \quad \frac{0+0}{1-1} = \frac{"0"}{0}$$

$$\text{L'H} = \lim_{\theta \rightarrow 0^+} \frac{1+2\theta}{-(-\sin \theta)}$$

$$= \lim_{\theta \rightarrow 0^+} \frac{1+2\theta}{\sin \theta} \quad [\text{den} = 0]$$

looks like  $\frac{1}{0}$  but b/c  $\theta \rightarrow 0^+$

$\sin \theta$  is +  $\Rightarrow$  limit is  $+\infty$

$$(e) \lim_{x \rightarrow 0} x^4 \sin \left( \frac{1}{x} \right)$$

note  $\sin \frac{1}{x}$  as  $x \rightarrow 0$  never 'settles down' but for all  $x$

$$-1 \leq \sin \left( \frac{1}{x} \right) \leq 1$$

$\Rightarrow$  if we mult the inequalities by  $x^4$

$$-x^4 \leq x^4 \sin \left( \frac{1}{x} \right) \leq x^4$$

$$\text{observe } \lim_{x \rightarrow 0} -x^4 = 0 = \lim_{x \rightarrow 0} x^4$$

so by the squeeze theorem

$$\lim_{x \rightarrow 0} x^4 \sin \left( \frac{1}{x} \right) = 0.$$

$$(b) \lim_{x \rightarrow \infty} \frac{x^2 + x - 2}{2x^2 - 8x + 6} \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x \rightarrow \infty} \frac{\frac{x^2}{x^2} + \frac{x}{x^2} - \frac{2}{x^2}}{2 - \frac{8x}{x^2} + \frac{6}{x^2}} = \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x} - \frac{2}{x^2}}{2 - \frac{8}{x} + \frac{6}{x^2}}$$

$$= \frac{\lim_{x \rightarrow \infty} (1 + \frac{1}{x} - \frac{2}{x^2})}{\lim_{x \rightarrow \infty} (2 - \frac{8}{x} + \frac{6}{x^2})} = \frac{1}{2}$$

$$(d) \lim_{x \rightarrow \infty} x \sin \left( \frac{5\pi}{x} \right) = \lim_{x \rightarrow \infty} \frac{\sin \left( \frac{5\pi}{x} \right)}{\frac{1}{x}} = \frac{0}{0}$$

$$\text{L'H} = \lim_{x \rightarrow \infty} \frac{\cos \left( \frac{5\pi}{x} \right) \cdot -\frac{5\pi}{x^2}}{-\frac{1}{x^2}}$$

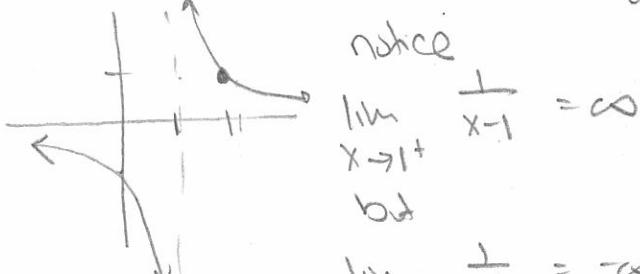
$$= \lim_{x \rightarrow \infty} \left[ \cos \left( \frac{5\pi}{x} \right) \cdot -\frac{5\pi}{x^2} \right] \div \left[ \frac{1}{x^2} \right]$$

$$= \lim_{x \rightarrow \infty} 5\pi \cos \left( \frac{5\pi}{x} \right) = 5\pi \lim_{x \rightarrow \infty} \cos \left( \frac{5\pi}{x} \right)$$

$$(f) \lim_{x \rightarrow 1} \frac{1}{x-1} \quad \begin{array}{l} \text{den} = 0 \\ 1-1=0 \end{array}$$

note  $\frac{1}{x-1}$  looks like the graph

of  $\frac{1}{x}$  shifted horiz. to the right 1 unit



notice

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty$$

but

$$\lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty$$

thus  $\lim_{x \rightarrow 1} \frac{1}{x-1}$  doesn't exist.

4. Let  $f(x) = \begin{cases} \sqrt{1 - (x+3)^2} & \text{if } -4 \leq x \leq -2 \\ 1 & \text{if } -2 < x < 2 \\ -(x-3)^2 + 2 & \text{if } 2 < x < 4 \\ -3 & \text{if } 4 < x \end{cases}$

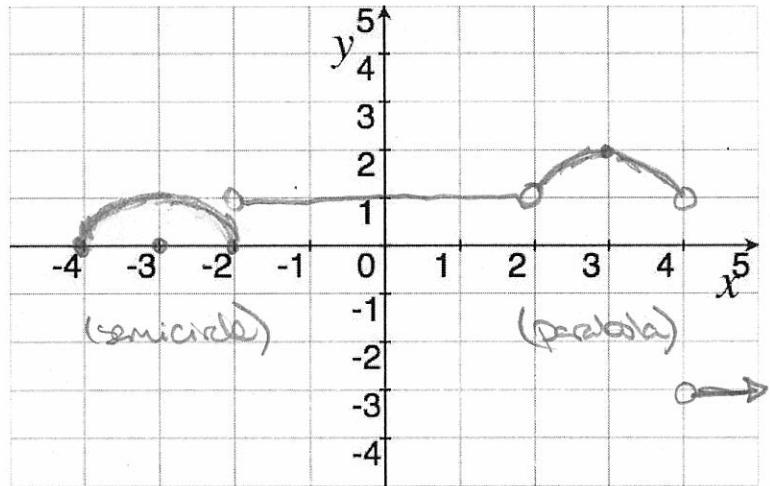
$y = \sqrt{1 - (x+3)^2} \Rightarrow y^2 + (x+3)^2 = 1$   
 circle centered at  $(-3, 0)$   
 with radius of 1.

Parabola w/ vertex  $(3, 2)$  open down

Graph  $f(x)$  and then graph  $f'(x)$  below on its own set of axes. Afterwards, answer the following questions.

(a)  $\lim_{x \rightarrow 2^-} f(x)$

1



(b)  $\lim_{x \rightarrow -2^+} f(x)$

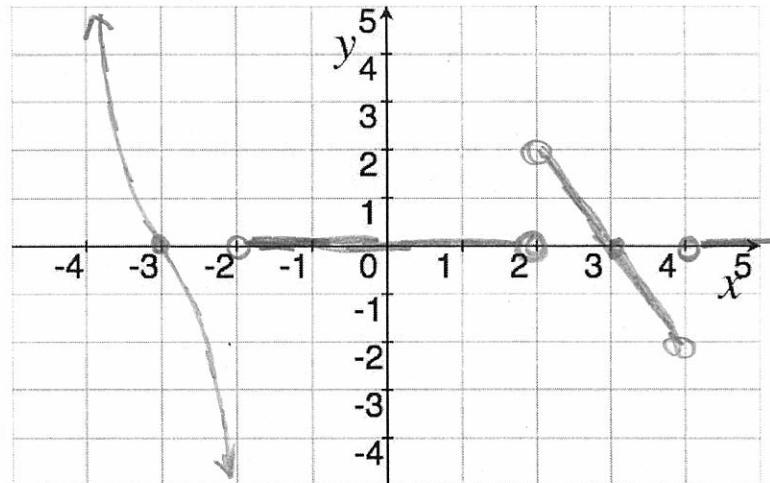
1

(c)  $\lim_{x \rightarrow 3} f'(x)$

0

(d)  $\lim_{x \rightarrow \infty} f(x)$

-3



5. Compute the derivatives of the following functions. You do *not* need to simplify.

$$(a) f(x) = x^3 + 3^x + \pi^\pi$$

$$f'(x) = 3x^2 + 3^x \ln 3 + 0$$

$$(b) g(t) = \ln(t) \left( \frac{2+t^2}{3t-1} \right)$$

$$\begin{aligned} g'(t) &= \ln(t) \left[ \frac{2+t^2}{3t-1} \right]' + [\ln(t)]' \left( \frac{2+t^2}{3t-1} \right) \\ &= \ln(t) \left[ \frac{(3t-1)(2t) - (2+t^2)(3)}{(3t-1)^2} \right] + \frac{1}{t} \left( \frac{2+t^2}{3t-1} \right) \end{aligned}$$

$$(c) h(\theta) = 7 \sec(\sqrt{\theta})$$

$$h'(\theta) = 7 \left[ (\cos(\theta^{\frac{1}{2}}))^{-1} \right]'$$

$$= 7 \cdot -1 (\cos(\theta^{\frac{1}{2}}))^{-2} \cdot (-\sin(\theta^{\frac{1}{2}})) \frac{1}{2} \theta^{-\frac{1}{2}}$$

$$= 7 \frac{\sin \sqrt{\theta}}{(\cos \sqrt{\theta})^2} \frac{1}{2} \frac{1}{\sqrt{\theta}} = \frac{7 \sin \sqrt{\theta}}{2\sqrt{\theta} (\cos \sqrt{\theta})^2}$$

$$(d) y = \sqrt{x} e^{x^7} (x^6 + 3)^{10} \quad \ln y = \ln [x^{\frac{1}{2}} e^{x^7} (x^6 + 3)^{10}]$$

$$\ln y = \frac{1}{2} \ln x + x^7 + 10 \ln(x^6 + 3) \quad \frac{dy}{dx}$$

$$\frac{1}{y} y' = \frac{1}{2} \cdot \frac{1}{x} + 7x^6 + \frac{10}{x^6 + 3} \cdot 6x^5$$

$$y' = y \left[ \frac{1}{2x} + 7x^6 + \frac{60x^5}{x^6 + 3} \right]$$

$$(c) y = (\cos(x))^x$$

$$\begin{aligned} \ln y &= \ln(\cos(x))^x \\ \ln y &= x \ln(\cos(x)) \end{aligned}$$

$$\frac{1}{y} y' = x [\ln(\cos(x))]' + (x)' \ln(\cos(x))$$

$$\frac{1}{y} y' = x \frac{1}{\cos(x)} \cdot -\sin(x) + 1 \cdot \ln(\cos(x))$$

$$y' = y \left[ -x \tan(x) + \ln(\cos(x)) \right]$$

$$(d) x^2 y^2 = 4 - y \arctan(5x) \quad \text{note: } [\arctan(x)]' = \frac{1}{1+x^2}$$

$$x^2 \frac{d}{dx} y + 2xy^2 = 0 - \left[ y \frac{1}{1+(5x)^2} \cdot 5 + y' \arctan 5x \right]$$

$$2x^2 y y' + 2xy^2 = -\frac{5y}{1+25x^2} - y' \arctan 5x$$

$$2x^2 y y' + y' \arctan 5x = -2xy^2 - \frac{5x}{1+25x^2}$$

$$y' (2x^2 y + \arctan 5x) = " "$$

$$4 \quad y' = \frac{-5x}{1+25x^2} - \frac{2xy^2}{2x^2 y + \arctan(5x)}$$

6. Find the equation of the line tangent to the graph of  $f$  when  $x = 2$  if  $f(x) = m(n(x))$ ,  $n(2) = -1$ ,  $m(-1) = 6$ ,  $n'(2) = 3$ , and  $m'(-1) = 5$ .

Looking for  $y = mx + b$

$m = f'(2)$

$f'(x) = (m \circ n)'(x)$  Chain Rule?  
 $= m'(n(x)) n'(x)$

So  $f'(2) = m'(n(2)) n'(2)$   
 $= m'(-1) \cdot 3 = 5 \cdot 3 = 15$

So we have  $y = 15x + b$ .  
Line passes through  $(2, f(2))$   
or  $(2, m(n(2))) = (2, m(-1)) = (2, 6)$   
so  
 $6 = 15 \cdot 2 + b \rightarrow b = 6 - 30 = -24$

Thus  
 $y = 15x - 24$

7. Find the antiderivative for each of the following functions:

(a)  $2x - x^3 + 7 \sin(x)$

$x^2 - \frac{1}{4}x^4 + 7 \cos(x)$

check:

$(x^2 - \frac{1}{4}x^4 + 7 \cos(x))' = 2x - \frac{1}{4} \cdot 4x^3 + 7(-\sin(x))$

off by negative sign in last term so

$x^2 - \frac{1}{4}x^4 - 7 \cos(x)$

(b)  $\frac{5 - 4x^3 + 2x^6}{x^6}$   
 $= \frac{5}{x^6} - \frac{4x^3}{x^6} + \frac{2x^6}{x^6}$   
 $= 5x^{-6} - 4x^{-3} + 2$

try  $-x^{-5} + x^{-2} + 2x$

check:

$(-x^{-5} + x^{-2} + 2x)' = -5x^{-6} - 2x^{-3} + 2$

off by neg sign & a factor of 2

$-x^{-5} + 2x^{-2} + 2x$

8. Consider the function  $f(x) = \sqrt[3]{x}$

(a) Evaluate the integral  $\int_1^8 \sqrt[3]{x} dx = F(8) - F(1)$  where  $F$  is an antider.

Note

$(\frac{3}{4}x^{\frac{4}{3}})' = \frac{3}{4} \cdot \frac{4}{3} x^{\frac{1}{3}} = x^{\frac{1}{3}}$  so  $\frac{3}{4}x^{\frac{4}{3}}$  is an antider.

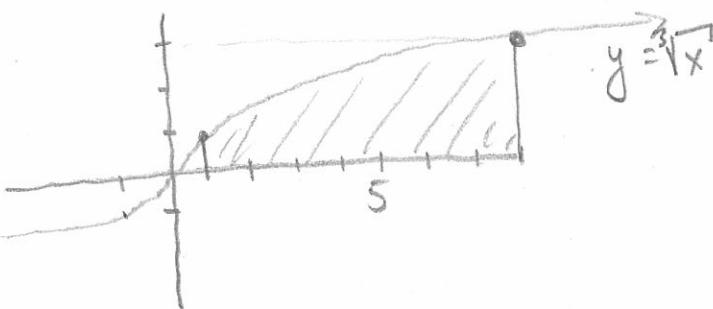
$\int_1^8 x^{\frac{1}{3}} dx = \frac{3}{4} x^{\frac{4}{3}} \Big|_1^8$   
 $= \frac{3}{4}(8)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}$

$= \frac{3}{4} \cdot 16 - \frac{3}{4}$

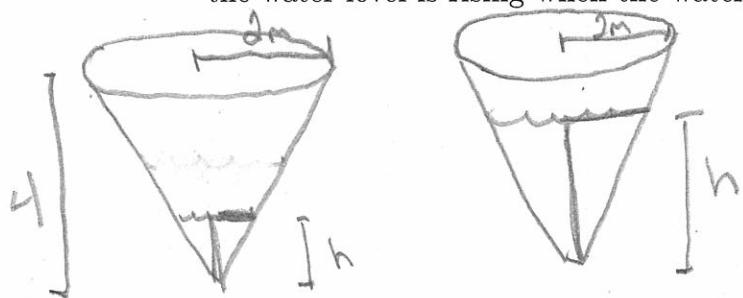
$= 12 - \frac{3}{4}$

$= \frac{48 - 3}{4}$

$= \frac{45}{4}$



9. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of  $2\text{m}^3/\text{min}$ , find the rate at which the water level is rising when the water is 3m deep.



want to find  $\frac{dh}{dt}$  |  
h=3m  
know  $\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$ .

V is volume of water      The volume of a cone is  $\frac{1}{3}\pi(\text{radius})^2 \cdot \text{height}$

$V = \frac{1}{3}\pi r^2 h$  this would be easier to take the derivative of if we could have it in terms of only 1 variable.

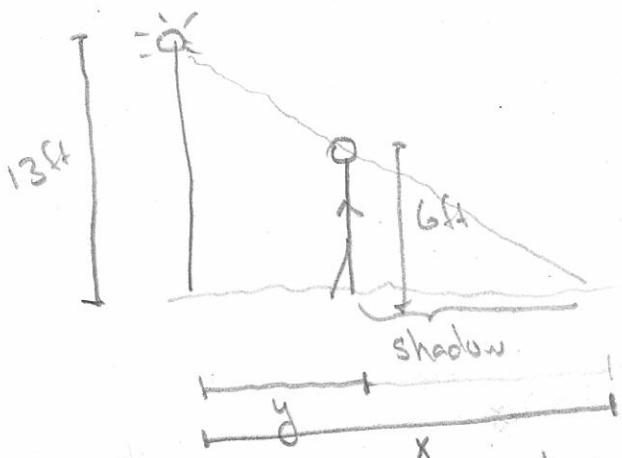
Note: similar triangles

$$\Rightarrow \frac{r}{h} = \frac{4}{4} \Rightarrow r = \frac{2h}{3}$$

$$V = \frac{1}{3}\pi h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{3} \cdot 3h^2 \cdot dh/dt$$

$$\Rightarrow \text{when } h=3 \quad \frac{dV}{dt} = \frac{\pi}{3} \cdot 3 \cdot 3^2 \cdot \frac{dh}{dt}$$

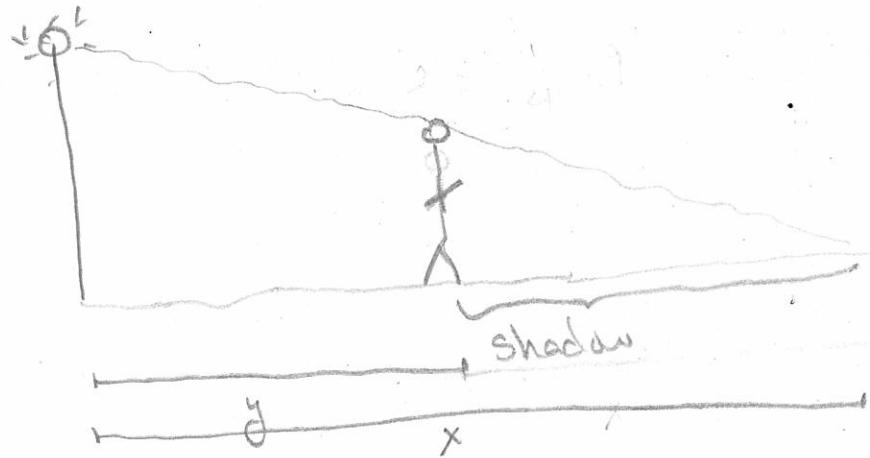
10. A street light is mounted at the top of a 13 ft pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 30 ft away from the pole?



$$\frac{dy}{dt} = 5 \text{ ft/s} \text{ want } \frac{dx}{dt}$$

Note similar triangles.

$$\frac{13}{x} = \frac{6}{y} \Rightarrow 13x - 13y = 6x \Rightarrow 7x = 13y$$

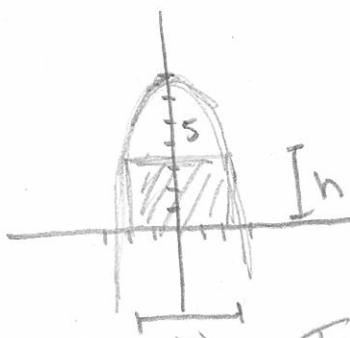


$$7 \frac{dx}{dt} = 13 \frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{13}{7} \frac{dy}{dt}$$

so when 30ft from the pole  
the tip of the shadow is  
moving at  $\frac{13}{7} \cdot 5 = \frac{65}{7} \text{ ft/s}$

11. Find the dimensions of the rectangle of largest area that has its base on the  $x$ -axis and its other two vertices above the  $x$ -axis and lying on the parabola  $y = 7 - x^2$



want to maximize area =  $w \cdot h$

$$\text{note } w = 2x$$

$$\text{and } h = y = 7 - x^2$$

$$\text{so area} = 2x(7 - x^2) = 14x - 2x^3$$

To maximize we need to find the extrema.

$$\begin{aligned} \text{Area}' &= 14 - 6x^2 \Rightarrow \frac{-14}{-6} = x^2 \quad \text{Area}(0) > \text{Area}'(0) \quad \text{max when} \\ 0 &= 14 - 6x^2 \qquad \qquad \qquad \Rightarrow x = \pm\sqrt{\frac{7}{3}} \quad -\sqrt{\frac{7}{3}} + \sqrt{\frac{7}{3}} - \\ -14 &= -6x^2 \end{aligned}$$

$$x = \sqrt{\frac{7}{3}}$$

$$\begin{aligned} \text{so width} &= 2\sqrt{\frac{7}{3}} \\ \text{and height} &= 7 - \frac{7}{3} \\ &= \frac{14}{3} \end{aligned}$$

12. A truck has a minimum speed of 9 mph in high gear. When traveling  $x$  mph, the truck burns diesel fuel at the rate of

$$0.003935 \left( \frac{675}{x} + x \right) \frac{\text{gal}}{\text{mile}}$$

Assume that the truck can not be driven over 63 mph, that diesel fuel costs \$2.84 a gallon, and that the driver is paid \$12 an hour. Find the speed that will minimize the cost of a 500 mile trip.

Total Cost = Cost of Gas + Cost of Driver.

$$\begin{aligned} &= 5.5877 \left( \frac{675}{x} + x \right) + \frac{6000}{x} \\ &= \frac{3771.6975}{x} + 5.5877x + \frac{6000}{x} \\ &= \frac{9771.6975}{x} + 5.5877x \end{aligned}$$

$$\text{Total cost}'(x) = \frac{-9771.6975}{x^2} + 5.5877$$

Cost of Gas:

$$\begin{aligned} &0.003935 \left( \frac{675}{x} + x \right) \cdot 500 \text{ miles} \cdot 2.84 \frac{\$}{\text{gal}} \\ &= 5.5877 \left( \frac{675}{x} + x \right) \text{ dollars} \end{aligned}$$

Cost of Driver:

$$\begin{aligned} &12 \frac{\$}{\text{hr}} \cdot \frac{500 \text{ miles}}{x \text{ miles}} \cdot 500 \text{ miles} \\ &= \frac{60000}{x} \text{ dollars} \end{aligned}$$

note  $-41.82 \text{ mph}$  won't work  
verify  $41.8 \text{ mph}$  is a min.

$$\frac{1}{\text{cost}'(1)} \frac{1}{41.8} \frac{1}{\text{cost}'(50)}$$

$$\Rightarrow 5.5877x^2 = 9771.6975$$

$$\Rightarrow x = \pm 41.82 \text{ mph}$$

$$\text{min} \rightarrow \rightarrow \rightarrow 41.82 \text{ mph}$$