

*Key*

Note: This is a practice midterm and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Let  $f$  and  $g$  be functions. Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

T (F)  $\frac{3x+y}{3z} = \frac{x+y}{z}$   $\frac{3(x+y)}{3z} = \frac{x+y}{z}$

T (F)  $(x+y)^2 = x^2 + y^2$   $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

T (F)  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  for all  $a$  only if  $\lim_{x \rightarrow a} g(x) \neq 0$

T (F) If  $\lim_{x \rightarrow \infty} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} g(x) = \infty$ , then  $\lim_{x \rightarrow \infty} [f(x) - g(x)] = 0$ .

T (F) If  $f$  is continuous at  $a$ , then  $f$  is differentiable at  $a$ .

(T) F If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

Let  $f(x) = x$   
and  $g(x) = x^2$   
 $f(x) - g(x) = x - x^2 = x(1-x)$   
which goes to  $-\infty$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. Find a formula for a function that has vertical asymptotes  $x = 2$  and  $x = 0$  and horizontal asymptote at  $y = -1$ .

$$\frac{-x^2+1}{x(x-2)}$$

~~not a cubic because it's not divisible by 3~~

Note: ~~and~~  $\lim_{x \rightarrow \infty} \frac{-x^2+1}{x^2-2x} = \lim_{x \rightarrow \infty} \frac{-1 + \frac{1}{x^2}}{1 - \frac{2}{x}} = -1$

Also

$\frac{-x^2+1}{x(x-2)}$	$-5.2$	$-50$	$49.7$	$4.7$
$x$	$.1$	$.01$	$-.01$	$-.1$

$\rightarrow \lim_{x \rightarrow 0^+} \frac{-x^2+1}{x(x-2)} = -\infty$  ↗

1  $\lim_{x \rightarrow 0^-} \frac{-x^2+1}{x(x-2)} = \infty$  ↘  
 $\therefore$  vert. asy.

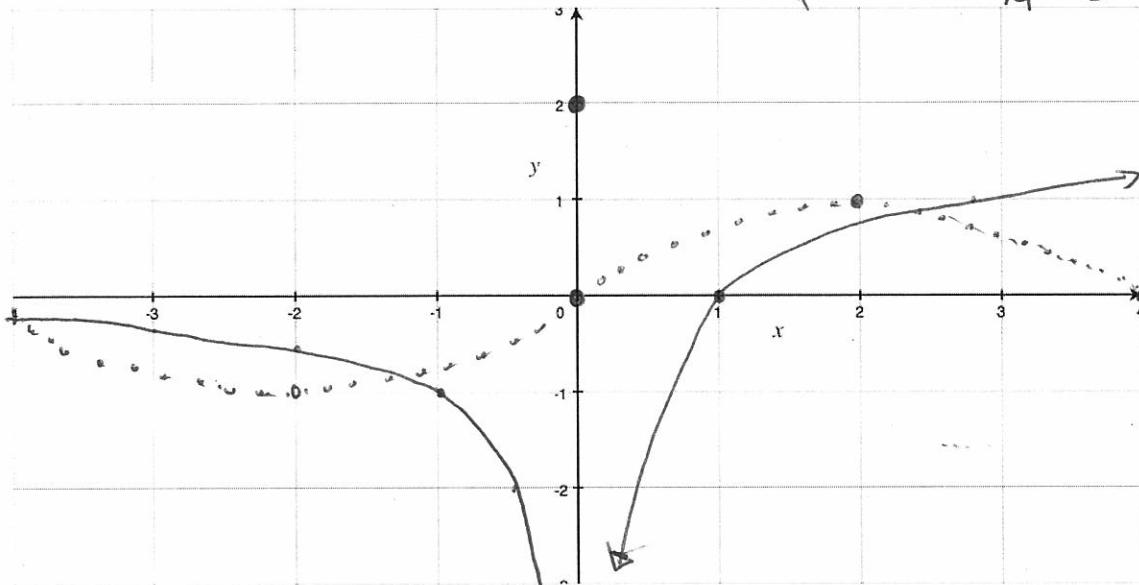
Similarly when  $x=2$ .

3. Given the rules of  $f$  and  $g$  below, graph both functions on the axis provided and evaluate the following

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0, \\ 2 & \text{if } x = 0, \\ \ln x & \text{if } x > 0, \end{cases} \quad (\text{solid})$$

$$(dotted) \quad g(x) = \sin\left(\frac{\pi}{4}x\right)$$

$$\text{period: } \frac{2\pi}{\frac{\pi}{4}} = 8$$



$$\lim_{x \rightarrow \infty} g(x)$$

$$\lim_{x \rightarrow 0} f(x)$$

$-\infty$

Does not exist.

$$f(0)$$

2

$$\lim_{x \rightarrow -2} (2f(x) + g(x))$$

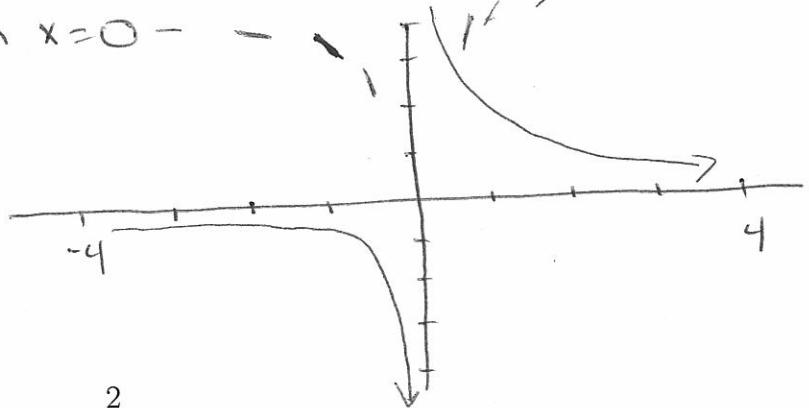
$$= 2 \lim_{x \rightarrow -2} f(x) + \lim_{x \rightarrow -2} g(x) = 2 \cdot \frac{1}{-2} + \sin\left(\frac{\pi}{4} \cdot -2\right) = -1 + 1 = 0$$

List any values that  $f$  is not continuous at:

not cont when  $x = 0$

○ horiz. tangent line

Graph  $g'(x)$



4. [] Find the limit if it exists, or explain why it does not exist.

$$\lim_{x \rightarrow 3} \frac{x^2 - 4x + 3}{x^2 - 2x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x-1)}{(x-3)(x+1)}$$

$$= \lim_{x \rightarrow 3} \frac{x-1}{x+1} = \frac{3-1}{3+1}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$\lim_{h \rightarrow 0} \frac{(1+h)^{-1} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h \rightarrow 0} \frac{\frac{1-(1+h)}{1+h}}{h}$$

$$= \lim_{h \rightarrow 0} \left[ \frac{-h}{1+h} \div h \right] = \lim_{h \rightarrow 0} \frac{-h}{h(1+h)}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{1+h} = \frac{-1}{1+0} = -1$$

$$\lim_{x \rightarrow 1} \frac{2x-2}{|x-1|} = \lim_{x \rightarrow 1} \frac{2(x-1)}{|x-1|}$$

if  $x > 1$

$$\lim_{x \rightarrow 1^+} \frac{2(x-1)}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{2(x-1)}{x-1}$$

$$= \lim_{x \rightarrow 1^+} 2 = 2$$

if  $x < 1$

$$\lim_{x \rightarrow 1^-} \frac{2(x-1)}{|x-1|} = \lim_{x \rightarrow 1^-} \frac{2(x-1)}{-(x-1)}$$

$$= \lim_{x \rightarrow 1^-} \frac{2}{-1} = -2$$

Does not exist b/c left limit doesn't equal the right.

$$\lim_{x \rightarrow \infty} e^{-2x} \sin x$$

Note  $-1 \leq \sin x \leq 1$

so  $e^{-2x} > 0$

$$-e^{-2x} \leq e^{-2x} \sin x \leq e^{-2x}$$

$$\text{Note } \lim_{x \rightarrow \infty} -e^{-2x} = 0$$

b/c graph transformations

$$\lim_{x \rightarrow \infty} e^{-2x} = 0 \text{ so}$$

$$\lim_{x \rightarrow \infty} \arctan(x^2 - x^4) \quad \lim_{x \rightarrow \infty} e^{-2x} \sin x = 0$$

$$\lim_{x \rightarrow \infty} \arctan(x^2(1-x^2))$$

$$= \arctan \lim_{x \rightarrow \infty} (x^2(1-x^2))$$

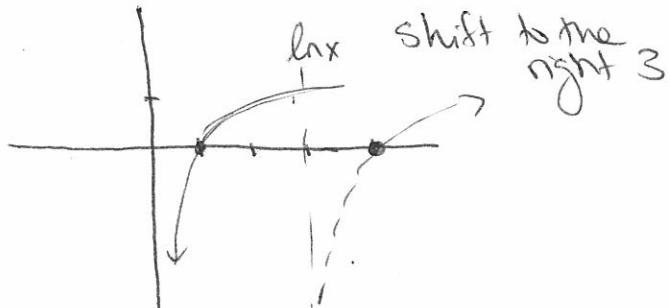
$$\text{and } \lim_{x \rightarrow \infty} x^2(1-x^2) = -\infty$$

By the graph of arctan then

$$\lim_{x \rightarrow \infty} \arctan(x^2 - x^4) = -\frac{\pi}{2}$$

$$x \rightarrow \infty$$

$$\lim_{x \rightarrow 3^+} \ln(x-3)$$



so  $-\infty$ .

5. Does  $f(x) = 2x^3 + 6x^2 - 10x - 30$  have a root between 2 and 3? Explain your reasoning and cite theorems if you use any.

Notice  $f(2) = 2 \cdot 2^3 + 6 \cdot 2^2 - 10 \cdot 2 - 30 = -10$

and  $f(3) = 2 \cdot 3^3 + 6 \cdot 3^2 - 10 \cdot 3 - 30 = 48$ .

The function  $f$  is a polynomial so ~~thus~~  $f$  is cont.  
This means to pass from  $(2, -10)$  to the point  $(3, 48)$ ,  
the graph of  $f$  must pass through the  $x$ -axis & thus  
have a root between 2 & 3. [Intermediate value Thm]

6. Find the equation for the line tangent to the graph of  $y = \frac{1}{(x-2)^2}$ , when  $x = 3$ .

looking for  $y = m \cdot x + b$ . Let  $g(x) = \frac{1}{(x-2)^2}$

$$m = \text{slope of line tangent} = g'(3) = \lim_{h \rightarrow 0} \frac{g(3+h) - g(3)}{h}$$

~~to~~  $\frac{1}{(x-2)^2}$  at  $x=3$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{(3+h-2)^2} - \frac{1}{(3-2)^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1}}{h}$$

$$\left\{ \lim_{h \rightarrow 0} \frac{1 - (1+h)^2}{(1+h)^2} \div h \right\} = \lim_{h \rightarrow 0} \frac{1 - 1 - 2h - h^2}{(1+h)^2} \cdot \frac{1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2h - h^2}{h(1+h)^2} = \lim_{h \rightarrow 0} \frac{h(-2-h)}{h(1+h)^2} = \frac{-2+0}{(1+0)^2} = -2$$

Notice the line passes through the point  $(3, g(3))$

[b/c it touches the graph of  $g$ ]

$$\text{or } (3, \frac{1}{(3-2)^2}) = (3, 1)$$

$$\text{so } 1 = -2(3) + b \Rightarrow 1 = -6 + b \Rightarrow b = 7$$

Thus  $y = -2x + 7$  works.

7. Suppose that the motion of a ball can be described by the equation  $f(t) = t^2 + t - 3$ .  
 Find the instantaneous velocity of the ball after 4 seconds.

velocity =  $\frac{\Delta \text{position}}{\Delta \text{time}}$  so inst. velocity =  $\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h}$

or rather

$$\lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{[(4+h)^2 + (4+h) - 3] - [4^2 + 4 - 3]}{h}$$

$f'(t) = 2t + 1$  by § 3.1 so  $f'(4) = 2 \cdot 4 + 1 = 8 + 1 = 9$  units/sec.

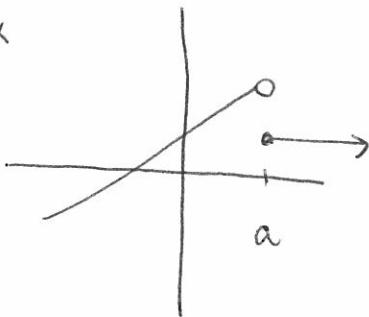
8. [] Using the definition, find the derivative of  $f(x) = \sqrt{2x - \frac{1}{2}}$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h) - \frac{1}{2}} - \sqrt{2x - \frac{1}{2}}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2x+2h-\frac{1}{2}} - \sqrt{2x-\frac{1}{2}}}{h} \frac{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})}{(\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}})} \\ &= \lim_{h \rightarrow 0} \frac{(2x+2h-\frac{1}{2}) - (2x-\frac{1}{2})}{h[\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}]} = \lim_{h \rightarrow 0} \frac{\frac{2h}{\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}}}{h} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2x+2h-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} \\ &= \frac{2}{\sqrt{2x+2 \cdot 0 - \frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} = \frac{2}{\sqrt{2x-\frac{1}{2}} + \sqrt{2x-\frac{1}{2}}} = \frac{2}{2\sqrt{2x-\frac{1}{2}}} = \frac{1}{\sqrt{2x-\frac{1}{2}}} \end{aligned}$$

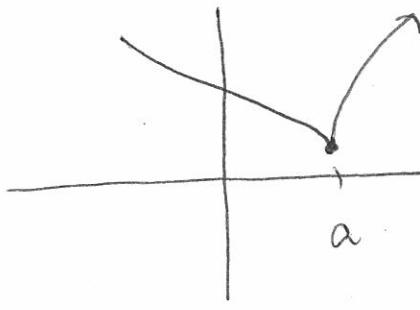
9. Describe 3 situations in which a function  $f(x)$  could **fail** to be differentiable at a point.

if:  $f$  is not cont.

ex



$f$  has a 'kink'



$f$  has a vert. tangent line

