Key

EXAM 2 — TMath 124 — Spring 2023

Show *all* your work (numerically, algebraically, or geometrically) for the following problems. Supporting work is needed to earn credit.

1. [2] (TrigAcitvity#2) Find $\lim_{x\to 0} \frac{\sin(2x)}{6x}$

algebraically: (.5) OR Graphically: (.5)

$\lim_{x\to 0} \frac{\sin(2x)}{6x} = \lim_{x\to 0} \frac{1}{3} \frac{\sin(2x)}{2x}$ (.5)

1 (.5)

$= \frac{1}{3}$

gut (.5)

2. Identify which derivative rule(s) you can use to find $\frac{dy}{dx}$. Do *not* find $\frac{dy}{dx}$!!

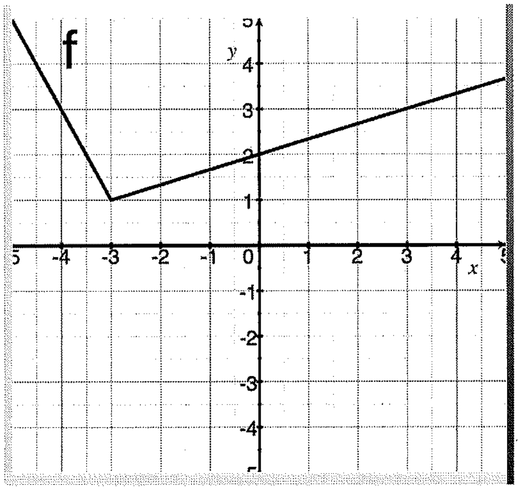
	Derivative Rule(s)
[2] (ImpExpActivity#5) $y = \sqrt{\frac{x-1}{x^4+1}}$ Start (.5)	logarithmic differentiation (1) (ln both sides) (use ln properties) (implicitly differentiate) and chain rule two times (.5) OR Chain rule (1) outside $u^{1/2}$ inside: (.5) (need quotient rule
[2] (§3.5 #24) $y + x\ln(y) = x^9$ start (.5)	differentiate each term use product rule on $x\ln(y)$ (.5) and implicit differentiation (1) with power rule in places OR algebra for $x\ln y = x^9 - y$ logarithmic differentiation (1) (ln both sides) (use ln properties) (implicitly differentiate) and power rule (.5)
[2] (WebHW8 #9) $y = e^{x^3-5x}$ Start (.5)	Chain Rule (1) outside e^u inside x^3-5x (.5) (power rule OR
[2] (WebHW10 #13) $y = (\tan(x))^x$ Start (.5)	logarithmic differentiation (1) (ln both sides) use properties to pull down x in the exponent (implicitly differentiate) with product rule (.5)

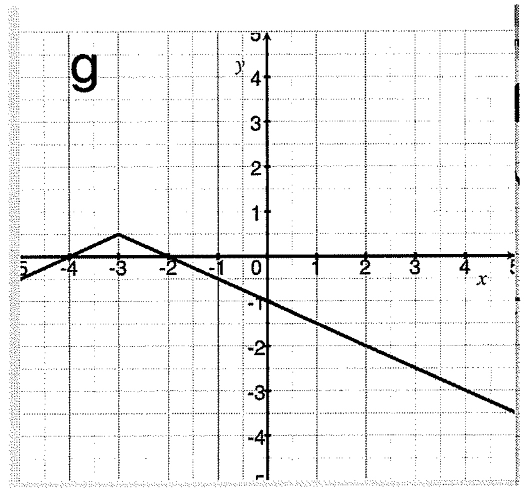
note: we cannot use power rule b/c x is an exponent.

note: we cannot use $\frac{d}{dx}(b^x)$ b/c the base depends on x

$\Rightarrow$ we have to use logarithmic differentiation

3. Use the graphs of f and g below for the following questions.





(a) [2] (ProductActivity#1) Find an x so that $g'(x)$ does not exist.

$x = -3$ note the corner when $x = -3$

(b) [3] (WebHW8#7) Estimate $\frac{d}{dx}(f(x)g(x))|_{x=0}$

$\frac{d}{dx}(f(x)g(x))\Big|_{x=0} = f(0)\frac{dg}{dx}\Big|_{x=0} + \frac{df}{dy}\Big|_{x=0} g(0)$

Product rule +.5
use right +.5

$= 2 \cdot \frac{-1}{2} + \frac{1}{3}(-1) = -1 - 1/3 = -4/3$

+.5 +.5 +.5 +.5

(c) [3] (Quiz3#1) If $c(x) = f(g(x))$, then estimate $c'(4)$.

$c'(4) = f'(g(4)) \cdot g'(4)$

$= f'(-3) \cdot (-1/2)$

+.5 +.5

Chain rule +.5
use right +.5
notation +.5

+.5 = DNE? corner @ $x = -3$ (see (a))

(d) [3] (§3.4#72) If $h(x) = g(3x - 1)$, then estimate $h'(2)$.

$h'(2) = g'(3(2)-1) \cdot \frac{d}{dx}(3x-1)\Big|_{x=2}$

Chain rule +.5
use right +.5
Composition +.5

$= g'(5) \cdot 3$

+.5 +.5

$= \frac{-1}{2} \cdot 3 = \frac{-3}{2}$

+.5

4. The differentiable functions f and g are defined for all real numbers. Values for f, f', g, and g' for various x values are given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	4	2	6
2	1	5	8	7
3	7	7	2	9

(a) [4] (PracticeExam#4) Given that $h(x) = \frac{f(x)}{3x+g(x)}$, find $h'(1)$.

quotient (+.5) did right (+.5) notation (+.5)

$h'(1) = \dfrac{(3(1)+g(1))f'(1) - f(1)[3+g'(1)]}{[3(1)+g(1)]^2}$ (+.5)

evaluating (+2) $= \dfrac{(3+2)4 - 3(3+6)}{(3+2)^2} = \dfrac{20-3(9)}{5^2} = \dfrac{20-27}{25} = \dfrac{-7}{25}$

(b) [3] (§3.10 #52) Find the linearization of f at $x = 2$.

notation (+.5)

Looking for $y - y_1 = m(x-x_1)$ (+.5)

m = slope of line tangent to f @ $x=2$

$= f'(2) = 5$ (+.5)

So $y - 1 = 5(x-2)$ (+.5) (+.5) Plug in (+.5)

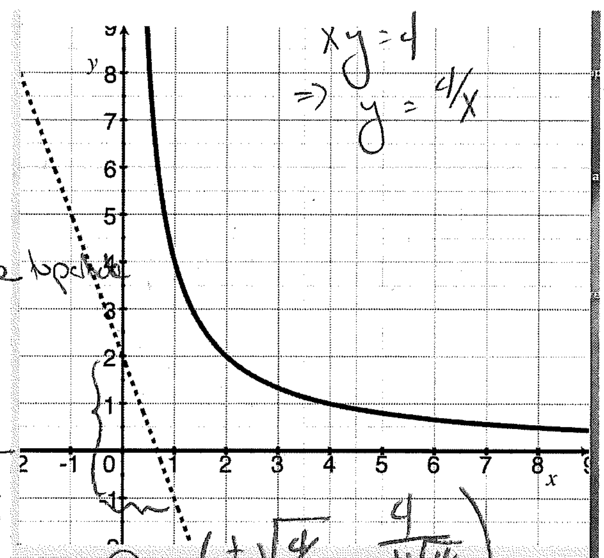
(c) [2] (§3.10 #52) Use the linearization of f to approximate $f(2.05)$.

notation (+.5)

ie plug into the line found in (b) (+1)

$5(2.05) - 9 = 1.25$ (+.5)

5. A particle moves along a hyperbola $xy = 4$ when $x > 0$. The graph is shown below with a solid curve. The dotted line is of a dust particle moving along a straight line.



$xy = 4$

$\Rightarrow y = 4/x$

(a) [4] (§3.2 #56) Find the point that the particle's movement is parallel to a dust particle moving along the dotted straight line graphed.

ie find x when

(+.5) slope of dotted line = slope of tang. line to particle

(+.5) $\frac{-3}{1} = y'(x)$ (+.5)

(+.5) note $y = 4/x \Rightarrow y' = -4x^{-2}$ (+1)

$-3 = -4x^{-2}$

$-3 = \frac{-4}{x^2}$

$\frac{-3x^2}{-3} = \frac{-4}{-3}$ $\quad x = \pm\sqrt{\frac{4}{3}}$

alg (+1)

so @ $\left(\pm\sqrt{4/3}, \frac{4}{\pm\sqrt{4/3}}\right)$

* (b) [4] (WebHW11#5) When the particle reaches an x value of 1, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

(+.5) $\left.\frac{dy}{dt}\right|_{x=1} = -3$ $\qquad$ WANT $\left.\frac{dx}{dt}\right|_{x=1}$ (+.5)

(+.5) $xy = 4$

$\frac{d}{dt}$

(+1) $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$

when $x=1$: $1(-3) + \left.\frac{dx}{dt}\right|_{x=1} \left.y\right|_{x=1} = 0$

(+.5) (+.5) from graph $y = 4$ (+.5)

$\Rightarrow -3 + \left.\frac{dx}{dt}\right|_{x=1} 4 = 0 \Rightarrow \left.\frac{dx}{dt}\right|_{x=1} = 3/4$

6. (RelatedActivity#2) On April 20, 2010 the Deepwater Horizon drilling rig exploded initiating the worst oil spill in US history. It was important to estimate the volume of oil spewing out each day, but it is hard to measure such a high volume flow directly. Instead one can measure the area of the spill from the air and the thickness of the spill and compute backwards. Suppose that the spilled oil is cylindrical in shape and a uniform thickness of 0.001m. On day 9 of the spill the area of the spill was 13,000,000,000 m^2 and the radius of the spill was increasing at a rate of 10 meters per day (Modified from Dr. Dawn's Blog)

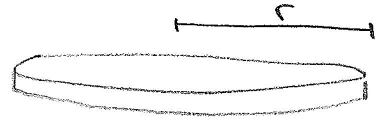
   (a) [3] Find a formula for how fast the volume is changing t days after the explosion.

   (b) [2] At what rate was the volume of the spill increasing on the 9th day?

   (c) [1] BP's original/official estimates of the flow rate were 160 to 790 m^3/day, how accurate were their estimates?

(+.5) solution

(a)



r } .001m

on day 9, Area = 13,000,000,000 m^2

$\left.\frac{dr}{dt}\right|_{t=9} = 10$ m/day.

Start (+.5)

(+.5) WANT $\frac{dV}{dt}$

$V = \pi r^2 \cdot \text{height}$ } (+.5) $= \pi r^2 (.001)$ (+.5)

$\frac{d}{dt}$ ↳ $\frac{dV}{dt} = .001\pi \cdot 2r \frac{dr}{dt}$ (+.5)

(b) WANT $\left.\frac{dV}{dt}\right|_{t=9}$ } (+.5)

$\frac{dV}{dt} = .001\pi \cdot 2\left(\sqrt{\frac{13{,}000{,}000{,}000}{\pi}}\right) \cdot 10$ } (+.5)

Area = 13,000,000,000

$\pi r^2 = 13{,}000{,}000{,}000$ } (+1)

$\Rightarrow r = \sqrt{\frac{13{,}000{,}000{,}000}{\pi}}$

$= 4{,}042$ m^3/day

(c) Not very accurate. very much an underestimate (+1)

(note official estimates that came later where a lot less than ours ≈ 3000 m^3/day)