Key

EXAM 1 — TMath 124 — Spring 2015

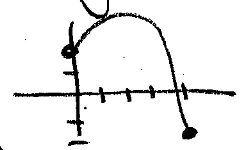
1. [7] TRUE/FALSE: Let f and g be functions. Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

T F $x + \frac{1}{x} = \frac{x^2+1}{x}$ $\frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + 1/x$

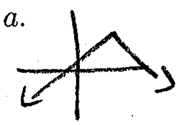
T F $\sqrt{x} = x^{-2}$ $\sqrt{x} = x^{1/2}$ $x^{-2} = 1/x^2$

T F If $\lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist. Consider $\lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$ (won't thm)

T F If f is a continuous function, $f(0) = 2$, and $f(4) = -2$, then $f(2)$ is between -2 and 2.
Could be higher or lower?



T F If f is continuous at a, then f is differentiable at a.



T F $(e^x)' = xe^{x-1}$. $(e^x)' = e^x$

T F $x^2 = 2x$. $(x^2)' = 2x$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

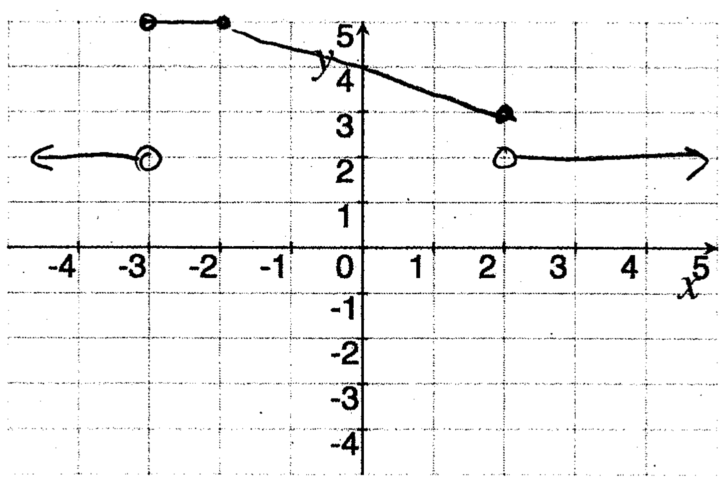
2. [5] (Quiz2 #2) Draw a graph for a function $\alpha(x)$, (+1) that satisfies all of the following:

(a) $\lim_{x\to\infty} \alpha(x) = 2$, (+1)

(b) α is not continuous at $x = -3$, (+1)

(c) $\alpha(2) = 3$ (+1)

(d) $\alpha'(0)$ is negative (+1)



Note: there are MANY correct answers

3. Let f be the piece-wise defined function:

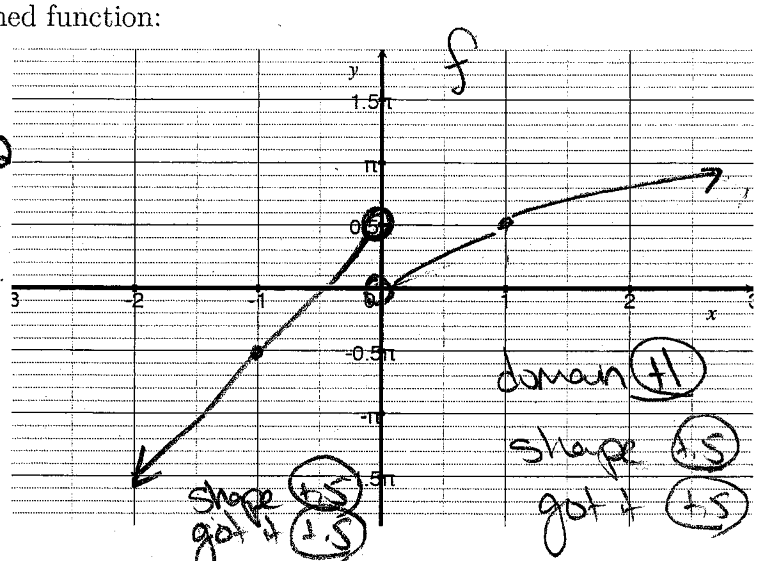
line w/ slope of π

$$f(x) = \begin{cases} \pi x + \frac{\pi}{2} & \text{if } x < 0, \\ 2\arctan(x) & \text{if } 0 < x, \end{cases}$$

vertical stretch by 2

(a) [3] (Quiz1 #1a) *Carefully* graph f on the axis provided.

(b) [3] (§2.8 #6) On the axis below, sketch the graph of $\frac{df}{dx}$.



(c) [6] Estimate the following, if possible: (WebHW1 #9, WebHW2 #1,WebHW4 #9, §2.3 #2)

$f(1)$ (+.5)

$2\arctan(1) = 2 \cdot \pi/4$

$= \pi/2$ (+.5)

$\lim_{x \to 0} f(x)$

DNE (+1)

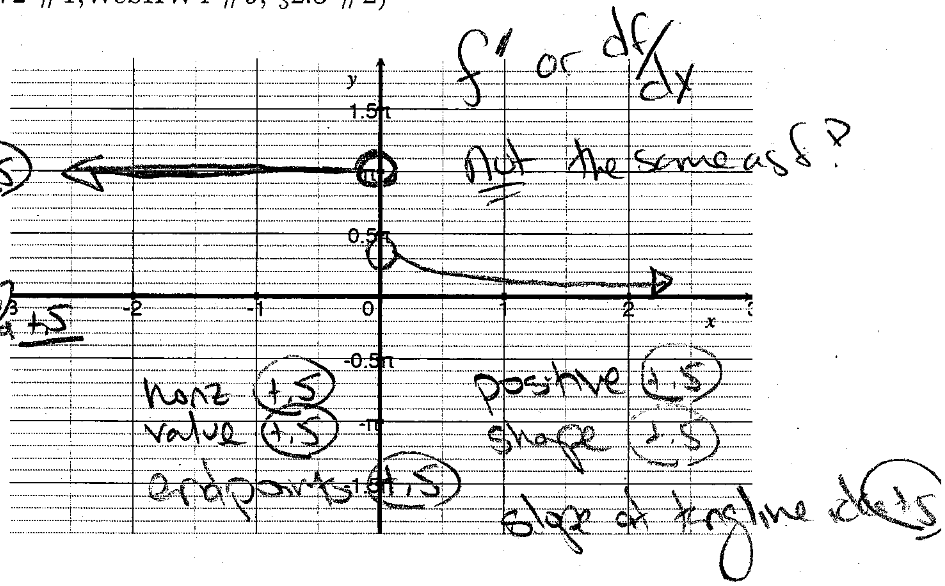
partial for 0 or $\pi/2$ +.5

$\lim_{x \to 0^-} f(x)$

$\pi/2$ (+1)

$\lim_{x \to \infty} f(x)$

π (+1)



$\lim_{x \to -2} (2f(x) - \pi) = 2\lim_{x \to -2} f(x) - \pi$ (+.5)

$2\left(-\frac{3}{2}\pi\right) - \pi$ (+.5) alg (+1)

$-3\pi - \pi = -4\pi$

questions about f?

4. [12] Find the limit if it exists, or explain why it does not exist.

(Lecture 4/7)

$$\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$$

notation (+.5)

note $-1 \le \sin \pi/x \le 1$

$\Rightarrow \quad -x^2 \le x^2 \sin \pi/x \le x^2$ (+1)

Since $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$ (+1)

by the squeeze thm (+1)

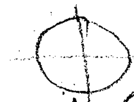
$\lim_{x\to 0} x^2 \sin(\pi/x) = 0$

or

x	$-.01$	$-.0001$	$.0001$	$.01$
$x^2 \sin \pi/x$				

table (+1) 2-sided (+1) notation (+.5) got it (+.5) alg (+.5)

(§2.5 #36)



$$\lim_{x\to\pi} \frac{2}{x-3} + \cos(x)$$

evaluation (+.5)

$\frac{2}{\pi-3} + \cos(\pi)$ by continuity (+1)

notation (+.5)

$\frac{2}{\pi-3} + -1 \approx 13.125$ got it (+.5)

or

x	3	3.14	3.16	3.5
$\frac{2}{x-3} + \cos x$				

got it (+.5) notation (+.5)

table (+.5) 2 sided (+1)

or

(+1) limit law 2, limit law 5, limit law 3, def of limit

(+.5) cont of cos

$\lim_{x\to\pi} \frac{2}{x-3} + \lim_{x\to\pi} \cos(x)$

$\frac{2}{\lim_{x\to\pi}(x-3)} + \lim_{x\to\pi} \cos(x)$

notation (+.5) got it (+.5)

(§2.6 #20)

$$\lim_{x\to\infty} \frac{x - x\sqrt{x}}{2x^{\frac{3}{2}} + 3x - 5}$$

exponent (+1)

leading power (+1)

notation (+.5)

got it (+.5)

sign (+.5)

$\lim_{x\to\infty} \frac{x - x x^{1/2}}{2x^{3/2} + 3x - 5} = \lim_{x\to\infty} \frac{x - x^{3/2}}{2x^{3/2} + 3x - 5}$

$= \lim_{x\to\infty} \frac{-x^{3/2}}{2x^{3/2}} = \lim_{x\to\infty} \frac{-1}{2} = -\frac{1}{2}$

or

$\lim_{x\to\infty} \frac{x - x^{3/2}}{2x^{3/2} + 3x - 5} \frac{(1/x^{3/2})}{(1/x^{3/2})} = \lim_{x\to\infty} \frac{\frac{1}{x^{1/2}} - 1}{2 + \frac{3}{x^{1/2}} - \frac{5}{x^{3/2}}}$

by the Big Little principal

$= -\frac{1}{2}$

exponents (+1)

notation (+.5)

factor of $1/x^{3/2}$ (+1)

got it (+.5)

alg (+.5)

or

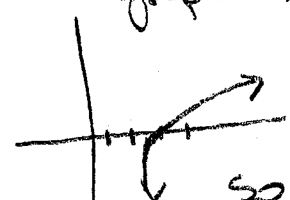
(practice #4)

$$\lim_{x\to 2^+} \ln(x-2)$$

(+.5) (+1)

graph of ln shifted right 2

$\ln(x-2)$



(+.5)

so $\lim_{x\to 2^+} \ln(x-2) = -\infty$

notation (+.5)

or

x	2.1	2.0001	2.000001
$\ln(x-2)$			

table (+1) 1 sided (+.5) got it (+.5) notation (+.5)

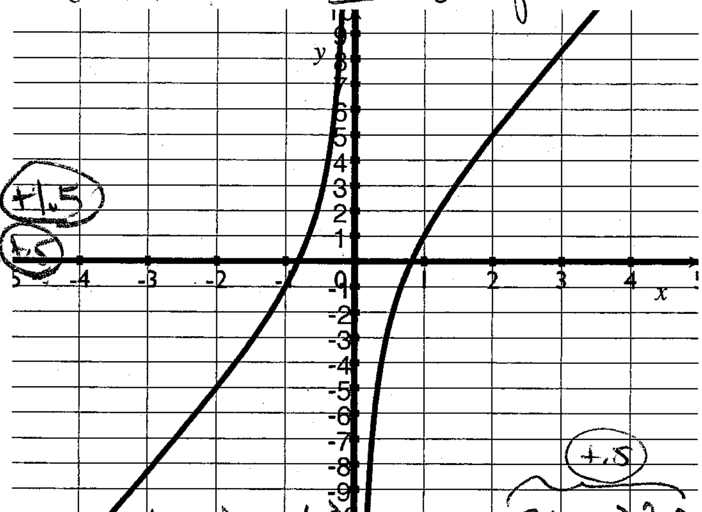
or

Partial +.5 plug in 2

+.5 eval $\ln(2-2)$

Partial: derivative of top +1
derivative of bottom +.5
quot rule +.5
use right +.5

5. Consider $g(x) = \dfrac{3x^2 - 2}{x}$ graphed to the right.



(a) [4] (PolyExp Wks #1) Find $\dfrac{dg}{dx}$

simplify +1.5
notation +.5

$$\frac{dg}{dx} = \frac{d}{dx}\left(\frac{3x^2-2}{x}\right)$$
$$= \frac{d}{dx}\left(\frac{3x^2}{x} - \frac{2}{x}\right)$$
$$= \frac{d}{dx}\left(3x - 2x^{-1}\right)$$
$$(+1) = \frac{d}{dx}(3x) - \frac{d}{dx}(2x^{-1})$$
$$(+1) = 3 + 2x^{-2}$$

or

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \; (+.5) = \lim_{h\to 0} \frac{\frac{3(x+h)^2-2}{x+h} - \frac{3x^2-2}{x}}{h} \; (+.5) \; (+.5)$$

notation +.5

fractions +.5
order of op +.5

$$= \lim_{h\to 0} \frac{(3(x+h)^2-2)x - (3x^2-2)(x+h)}{x(x+h)} \div h = \lim_{h\to 0} \frac{(3(x^2+2xh+h^2)-2)x - (3x^2-2)(x+h)}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{(\cancel{3x^3}+6x^2h+3h^2x-\cancel{2x}) - (\cancel{3x^3}+3x^2h-\cancel{2x}-2h)}{hx(x+h)} = \lim_{h\to 0} \frac{\cancel{h}(6x^2+3h-3x^2-2)}{\cancel{h}(x+h)x}$$

eval limit (+1)

$$= \lim_{h\to 0} \frac{3x^2+3h-2}{(x+h)x} = \frac{3x^2-2}{x^2}$$

(b) [2] Find $g'(1)$

$$g'(1) = \left.\frac{dg}{dx}\right|_{x=1} = 3 + 2(1)^{-2} = 3+2 = 5$$

(+1) plug in 1 to g'
got it (+1)

or

$$\lim_{h\to 0} \frac{g(1+h)-g(1)}{h} = \lim_{h\to 0} \frac{\frac{3(1+h)^2-2}{1+h} - 1}{h} = \ldots$$

(c) [3] (Quiz2 #3) Find the equation of the line tangent to g when $x = 1$.

Looking for $y = mx + b$ (+.5)

$m =$ slope of line tangent to g when $x=1$ } (+.5)

$= g'(1)$

$= 5$ from part (b) (+.5)

passes thru $(1, g(1)) = \left(1, \frac{3\cdot 1^2 - 2}{1}\right) = (1,1)$ (+.5)

$y - 1 = 5(x-1)$ (+1) or ((+1) $1 = 5(1) + b$, $-4 = b$) So $y = 5x - 4$

6. Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) [5] (Story Wks #6) The shuttle Discovery launched the Hubble Space Telescope April 24th 1990.
The shuttle's distance traveled from liftoff ($t = 0$) to jettisoning the rocket boosters ($t = 126$s) was well modeled by the function:

$$0.0003255t^4 - 0.03009667t^3 + 11.805t^2 - 3.083t$$

i. [2] Find a function that describes the velocity of the shuttle.
ii. [1] Find a function that describes the acceleration of the shuttle.
iii. [2] Identify when the maximum acceleration is obtained by the shuttle in the first 126 seconds.

(b) [5] (PracticeExam) The total cost of repaying a student loan at an interest rate of $r\%$ per year is $C = f(r)$.

i. [2] What is the meaning of the derivative $f'(r)$? What are its units?
ii. [1] What does the statement $f'(10) = 1200$ mean?
iii. [2] Is $f'(r)$ always positive or does it change sign? Justify your answer.

a) start (+.5)

i) velocity $= \frac{d}{dt}(\text{distance})$ (+.5)

$= \frac{d}{dt}(.0003255t^4 - .03009667t^3 + 11.805t^2 - 3.083t)$

piece by piece (+.5) $= 4 \cdot .0003255\,t^3 - 3 \cdot .03009667t^2 + 2 \cdot 11.805\,t - 3.082$

power rule (+.5) $= .001302t^3 - .090290\,t^2 + 23.61t - 3.082$

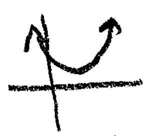
ii) acceleration $= \frac{d}{dt}(\text{velocity})$ (+.5)

power rule (+.5) $= 3 \cdot .001302\,t^2 - 2 \cdot .090290\,t + 23.61$

$= .003906t^2 - .180590\,t + 23.61$

iii) Note acceleration is a parabola

justification (+1) facing up



notation (+.5) so max happens @ the end points (at $t = 0$ or $t = 126$)

acceleration @ $t = 0$ is 23.61

@ $t = 126$ is 62.87

So max at $t = 126$ (+.5)

b) start (+.5)

(+1) cost of repaying loan / change in interest rate

i) $f'(r) = \lim\limits_{\text{change in } r \to 0}$

(+.5) units \$/%

ii) $f'(10) = 1200$ means increasing the interest rate from 10% (+.5) will increase the total cost of the loan w/ an instantaneous (+.5) rate of \$1200/%

iii) $f'(r)$ will always be positive (+.5) since increasing the interest rate will always increase the total cost of a loan.

justification (+1) notation (+.5)