FINAL **TMath 124** Practice

Key

Note: This is a practice final and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. [] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be differentiable functions and h be a constant.

   T F $\frac{x+h}{2x} = \frac{1+h}{x}$ $\quad \frac{1+h}{x} = \frac{2(1+h)}{2x} = \frac{2+2h}{2x}$

   T F $\sqrt{x^2+h^2} = x+h$ $\quad$ let $x=1$ and $h=1$ note $\sqrt{1^2+1^2} \neq 1+1$

   T F $\lim_{x\to r} f(x) = f(r)$ for all r in the domain of f. $\quad$ we were told that f is differentiable thus f is continuous.

   T F If $\lim_{x\to r} g(x) = 0$, then $\lim_{x\to r} \frac{f(x)}{g(x)}$ does not exist. $\quad$ let $f(x) = (x+1)(x-1)$, $g(x) = x-1$ and consider $\lim_{x\to 1} \frac{f(x)}{g(x)} = 2$

   T F $\frac{d}{dx}\left(\frac{1}{x}\right) = -1$ $\quad \frac{d}{dx}(x^{-1}) = -x^{-2} = -1/x^2$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [] Sketch the graph and then *find the formula* of an example function f that satisfies the following conditions:

   (a) $f(2) = 2$

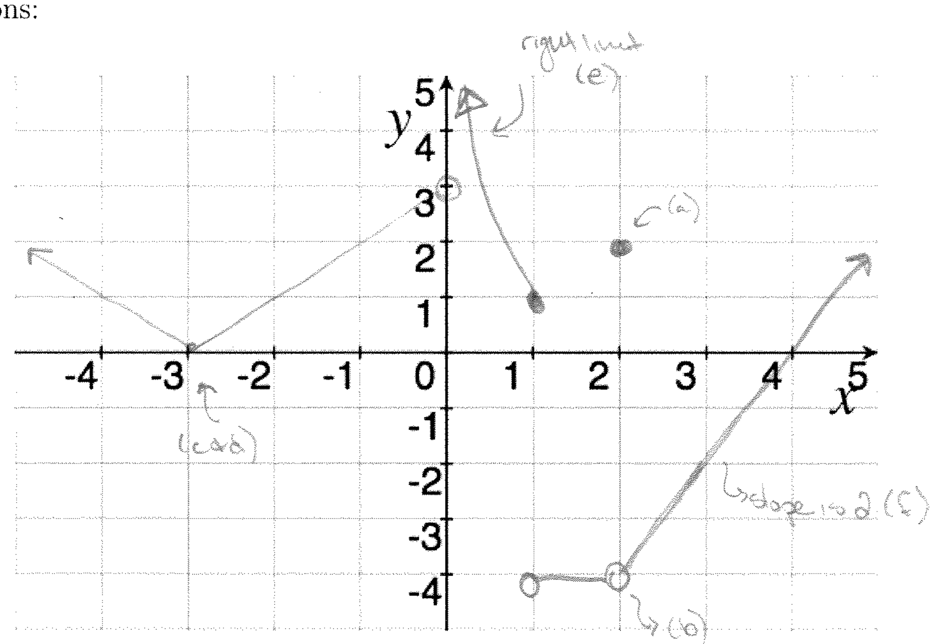
   (b) $\lim\limits_{x\to 2} f(x) = -4$

   (c) f is not differentiable when $x = -3$

   (d) f is continuous when $x = -3$

   (e) $\lim\limits_{x\to 0^+} f(x) = \infty$

   (f) $f'(4) = 2$



$$f(x) = \begin{cases} |x+3| & \text{if } x<0 \\ \frac{1}{x} & \text{if } 0<x\le 1 \\ -4 & \text{if } 1<x<2 \\ 2 & \text{if } x=2 \\ 2x-8 & \text{if } 2<x \end{cases}$$

3. Compute the following limits:

(a) $\lim_{x\to 1} \dfrac{x^2+x-2}{2x^2-8x+6}$ $\left[\begin{array}{l} \text{den} = 0 \\ 2-8+6=0 \end{array}\right]$

$= \lim_{x\to 1} \dfrac{(x+2)\cancel{(x-1)}}{2(x-3)\cancel{(x-1)}}$

$= \lim_{x\to 1} \dfrac{x+2}{2(x-3)} = \dfrac{1+2}{2(1-3)} = \dfrac{3}{2\cdot -2}$

$= \dfrac{3}{-4}$

(b) $\lim_{x\to\infty} \dfrac{x^2+x-2}{2x^2-8x+6}$ $\dfrac{1/x^2}{1/x^2}$

$\lim_{x\to\infty} \dfrac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{2}{x^2}}{2\frac{x^2}{x^2}-8\frac{x}{x^2}+\frac{6}{x^2}} = \lim_{x\to\infty} \dfrac{1+\frac{1}{x}-\frac{2}{x^2}}{2-\frac{8}{x}+\frac{6}{x^2}}$

$= \dfrac{\lim_{x\to\infty}(1+\cancel{\frac{1}{x}}^{\,0}-\cancel{\frac{2}{x^2}}^{\,0})}{\lim_{x\to\infty}(2-\underbrace{\cancel{\frac{8}{x}}}_{0}+\underbrace{\cancel{\frac{6}{x^2}}}_{0})} = \dfrac{1}{2}$

(c) $\lim_{\theta\to 0^+} \dfrac{\theta+\theta^2}{1-\cos\theta}$ $\dfrac{0+0}{1-1} = \text{“}\dfrac{0}{0}\text{”}$

$\overset{L'H}{=} \lim_{\theta\to 0^+} \dfrac{1+2\theta}{-(-\sin\theta)}$

$= \lim_{\theta\to 0^+} \dfrac{1+2\theta}{\sin\theta}$ $\left[\begin{array}{l} \text{den} = 0 \\ \sin 0 = 0 \end{array}\right]$

looks like "$\frac{1}{0}$" but b/c $\theta\to 0^+$

$\sin\theta$ is + $\Rightarrow$ limit is $+\infty$

(d) $\lim_{x\to\infty} x\sin\left(\dfrac{5\pi}{x}\right) = \lim_{x\to\infty} \dfrac{\sin(5\pi/x)}{1/x}$ "$\dfrac{0}{0}$"

$\overset{L'H}{=} \lim_{x\to\infty} \dfrac{\cos(5\pi/x)\cdot\frac{-5\pi}{x^2}}{-1/x^2}$

$= \lim_{x\to\infty} \left[\cos(5\pi/x)\cdot\dfrac{-5\pi}{x^2}\right] \div \left[\dfrac{-1}{x^2}\right]$

$= \lim_{x\to\infty} 5\pi\cos(5\pi/x) = 5\pi \lim_{x\to\infty} \cos\left(\frac{5\pi}{x}\right)$

$= 5\pi\cos(0) = 5\pi$

(e) $\lim_{x\to 0} x^4\sin\left(\dfrac{1}{x}\right)$

note $\sin 1/x$ as $x\to 0$ never 'settles down' but for all x

$-1 \le \sin(1/x) \le 1$

$\Rightarrow$ if we mult the inequalities by x^4

$-x^4 \le x^4\sin(1/x) \le x^4$

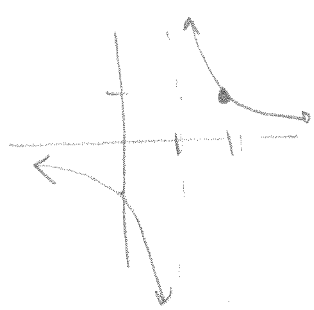
Observe $\lim_{x\to 0} -x^4 = 0 = \lim_{x\to 0} x^4$

So by the squeeze theorem

$\lim_{x\to 0} x^4\sin\left(\frac{1}{x}\right) = 0.$

(f) $\lim_{x\to 1} \dfrac{1}{x-1}$ $\left[\begin{array}{l} \text{den} = 0 \\ 1-1=0 \end{array}\right]$

note $\dfrac{1}{x-1}$ looks like the graph of $1/x$ shifted horiz. to the right 1 unit



notice

$\lim_{x\to 1^+} \dfrac{1}{x-1} = \infty$

but

$\lim_{x\to 1^-} \dfrac{1}{x-1} = -\infty$

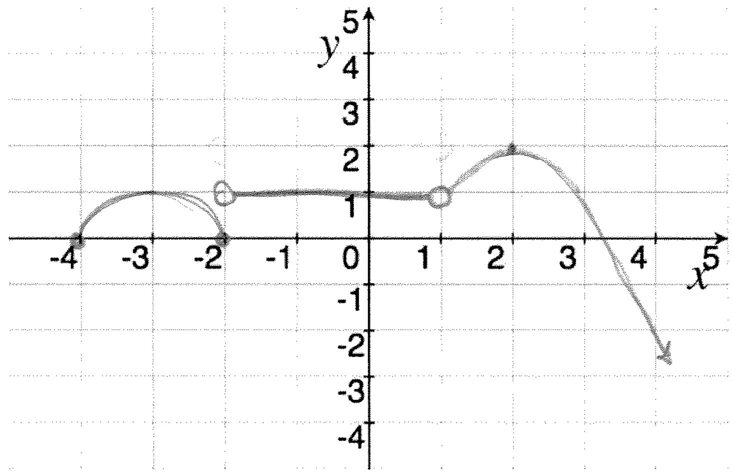
Thus $\lim_{x\to 1} \dfrac{1}{x-1}$ doesn't exist.

4. Let $f(x) = \begin{cases} \sqrt{1-(x+3)^2} & \text{if } -4 \le x \le -2 \\ 1 & \text{if } -2 < x < 1 \\ -(x-2)^2+2 & \text{if } 1 < x \end{cases}$

$y=\sqrt{1-(x+3)^2} \Rightarrow y^2+(x+3)^2=1$ circle centered at $(-3,0)$ with radius 1

parabola with vertex $(2,2)$ opening down

Graph $f(x)$ and then sketch the graph $f'(x)$ below on its own set of axes. Afterwards, answer the following questions.



(a) $\lim_{x\to 1} f(x)$

1

(b) $\lim_{x\to 3}[4f(x)-7]$

$4\lim_{x\to 3} f(x) - 7 = 4\cdot 1 - 7 = -3$

(c) $\lim_{x\to -2} f(x)$

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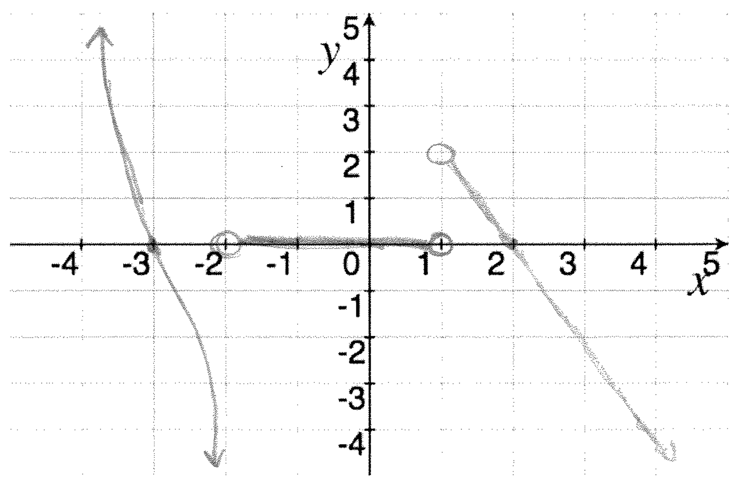
(d) $\lim_{x\to -2^-} f(x)$

0



(e) $\lim_{x\to 3} f'(x)$

-2 (using graph of f')

(f) $\lim_{x\to\infty} f(x)$

$-\infty$

(g) $[f+f]'(2)$

$= f'(2) + f'(2)$

$= 0 + 0$

$= 0$

note

$y^2+(x+3)^2=1$

$\frac{d}{dx}(y^2+(x+3)^2) = \frac{d}{dx}(1)$

$2yy' + 2(x+3)(1) = 0$

$y' = \frac{-2(x+3)}{2y}$

$y' = \frac{-(x+3)}{\sqrt{1-(x+3)}}$

note

$\frac{d}{dx}(-(x-2)^2+2)$

$\frac{d}{dx}(-x^2+4x-2)$

$-2x+4$

5. Compute the derivatives of the following functions. You do *not* need to simplify.

(*a*) $f(x) = x^3 + 3^x + \pi^\pi$

$f'(x) = 3x^2 + 3^x \ln 3 + 0$

(*b*) $g(t) = \ln(t)\left(\frac{2+t^2}{3t-1}\right)$

$g'(t) = \ln(t)\left[\frac{2+t^2}{3t-1}\right]' + [\ln(t)]'\left(\frac{2+t^2}{3t-1}\right)$

$= \ln(t)\left[\frac{(3t-1)(2t) - (2+t^2)(3)}{(3t-1)^2}\right] + \frac{1}{t}\left(\frac{2+t^2}{3t-1}\right)$

(*c*) $h(\theta) = 7\sec(\sqrt{\theta})$

$h'(\theta) = 7\left[\left(\cos(\theta^{1/2})\right)^{-1}\right]'$

$= 7 \cdot -1\left(\cos(\theta^{1/2})\right)^{-2} \cdot \left(-\sin(\theta^{1/2})\right)\frac{1}{2}\theta^{-1/2}$

$= 7\frac{\sin\sqrt{\theta}}{(\cos\sqrt{\theta})^2}\frac{1}{2}\frac{1}{\sqrt{\theta}} = \frac{7\sin\sqrt{\theta}}{2\sqrt{\theta}(\cos\sqrt{\theta})^2}$

(*d*) $y = \sqrt{x}e^{x^7}(x^6+3)^{10}$ $\ln y = \ln\left[x^{1/2}e^{x^7}(x^6+3)^{10}\right]$

$\ln y = \frac{1}{2}\ln x + x^7 + 10\ln(x^6+3)$ $\frac{d}{dx}$

$\frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 7x^6 + \frac{10}{x^6+3}\cdot 6x^5$

$y' = y\left[\frac{1}{2x} + 7x^6 + \frac{60x^5}{x^6+3}\right]$

(*c*) $y = (\cos(x))^x$

$\ln y = \ln(\cos(x))^x$

$\ln y = x\ln(\cos(x))$

$\frac{d}{dx}$ (

$\frac{1}{y}y' = x[\ln(\cos(x))]' + (x)'\ln(\cos(x))$

$\frac{1}{y}y' = x\frac{1}{\cos(x)}\cdot -\sin(x) + 1\cdot\ln(\cos(x))$

$y' = y\left[-x\tan(x) + \ln(\cos(x))\right]$

(*d*) $x^2y^2 = 4 - y\arctan(5x)$ note: $[\arctan(x)]' = \frac{1}{1+x^2}$

$x^2 2yy' + 2xy^2 = 0 - \left[y\frac{1}{1+(5x)^2}\cdot 5 + y'\arctan 5x\right]$

$2x^2yy' + 2xy^2 = \frac{-5y}{1+25x^2} - y'\arctan 5x$

$2x^2yy' + y'\arctan 5x = -2xy^2 - \frac{5x}{1+25x^2}$

$y'(2x^2y + \arctan 5x) = $ " "

$y' = \frac{\frac{-5x}{1+25x^2} - 2xy^2}{2x^2y + \arctan(5x)}$

6. Find the equation of the line tangent to the graph of f when $x = 2$ if $f(x) = m(n(x))$, $n(2) = -1$, $m(-1) = 6$, $n'(2) = 3$, and $m'(-1) = 5$.

Looking for $y = mx + b$

$m = f'(2)$

$f'(x) = (m \circ n)'(x)$ Chain Rule?

$= m'(n(x))\, n'(x)$

So $f'(2) = m'(n(2))\, n'(2)$

$= m'(-1) \cdot 3 = 5 \cdot 3 = 15$

So we have $y = 15x + b$.

Line passes through $(2, f(2))$

or $(2, m(n(2))) = (2, m(-1)) = (2, 6)$

So

$6 = 15 \cdot 2 + b \rightarrow b = 6 - 30 = -24$

Thus

$y = 15x - 24$

7. Find the antiderivative for each of the following functions:

(a) $2x - x^3 + 7\sin(x)$

$x^2 - \frac{1}{4}x^4 + 7\cos(x)$

Check:

$(x^2 - \frac{1}{4}x^4 + 7\cos(x))' = 2x - \frac{1}{4} \cdot 4x^3 + 7(-\sin(x))$

off by negative sign in last term so

$x^2 - \frac{1}{4}x^4 - 7\cos(x)$

(b) $\dfrac{5 - 4x^3 + 2x^6}{x^6}$ $= \dfrac{5}{x^6} - \dfrac{4x^3}{x^6} + \dfrac{2x^6}{x^6}$

$= 5x^{-6} - 4x^{-3} + 2$

try $x^{-5} + x^{-2} + 2x$

check:

$(x^{-5} + x^{-2} + 2x)' = -5x^{-6} - 2x^{-3} + 2$

off by neg sign & a factor of 2

$-x^{-5} + 2x^{-2} + 2x$

8. Consider the function $f(x) = \sqrt[3]{x}$

(a) Evaluate the integral $\displaystyle\int_1^8 \sqrt[3]{x}\, dx$ $= F(8) - F(1)$ where F is an antider.

note

$\left(\frac{3}{4}x^{4/3}\right)' = \frac{3}{4} \cdot \frac{4}{3} x^{1/3} = x^{1/3}$ so $\frac{3}{4}x^{4/3}$ is an antider.

$\displaystyle\int_1^8 x^{1/3}\, dx = \frac{3}{4}x^{4/3}\Big]_1^8$

$= \frac{3}{4}(8)^{4/3} - \frac{3}{4}(1)^{4/3}$

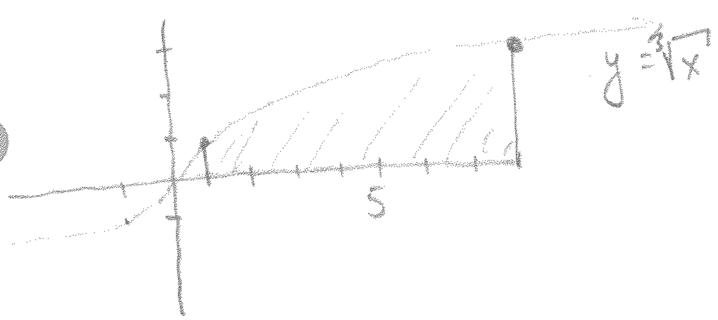
$= \frac{3}{4} \cdot 2^4 - \frac{3}{4}$

$= 12 - \frac{3}{4}$

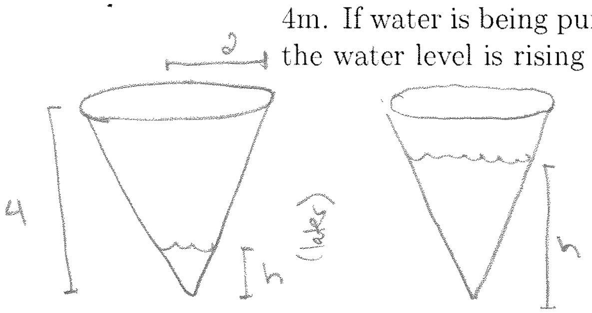
$= \frac{48 - 3}{4}$

$= \frac{45}{4}$

(b) Draw a picture that corresponds to the area you computed in (a).



9. A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3m deep.



Want to find $\left.\frac{dh}{dt}\right|_{h=3m}$

Know $dV/dt = 2\ m^3/min$

we need to find a relation between V & h

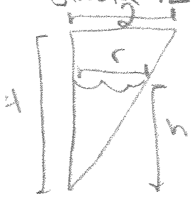
Let V be the volume of the water. Recall the volume of a cone is $\frac{1}{3}\pi(\text{radius})^2 \cdot \text{height}$

$V = \frac{\pi}{3} r^2 h$ it would be easier to take the derivative if we had only 1 variable

Relation between r & h

Similar Δ's $\Rightarrow \frac{2}{r} = \frac{4}{h}$

$\Rightarrow r = \frac{2h}{4} = \frac{h}{2}$



So $V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h$

$= \frac{\pi}{12} h^3$

So $dV/dt = \frac{d}{dt}\left(\frac{\pi}{12} h^3\right)$

$dV/dt = \frac{\pi}{12} \cdot 3h^2 \, dh/dt$

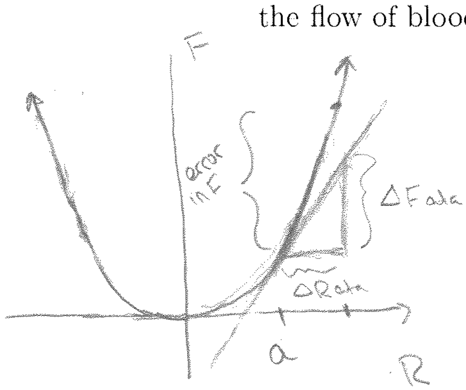
when $h = 3$

$2 = \frac{\pi}{12} \cdot 3 \cdot 3^2 \left.\frac{dh}{dt}\right|$

$\Rightarrow dh/dt = \frac{8}{9\pi}$

10. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel: $F = kR^4$. A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Use a linear approximation to show that the relative change in F is about four times the relative change in R. Then approximate how a 5% increase in the radius will affect the flow of blood?



Let a be the radius of a blood vessel,

we want to find $\frac{\text{error in F at a}}{\text{F at a}}$

which is well approximated by $\frac{\Delta F \text{ at a}}{\text{F at a}}$.

Use the notation shown in the picture above

Slope of line tangent to F at a $= \frac{\text{rise}}{\text{run}} = \frac{\Delta F \text{ at a}}{\Delta R \text{ at a}}$

$\Rightarrow \left. F' \right|_{R=a} = \frac{\Delta F \text{ at a}}{\Delta R \text{ at a}}$

note $F' = 4KR^3$

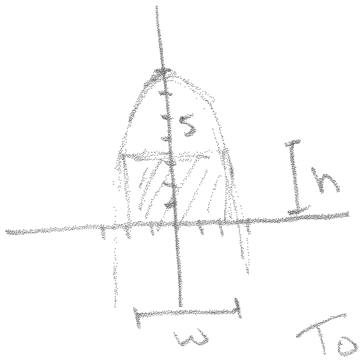
$4Ka^3 = \frac{\Delta F \text{ at a}}{\Delta R \text{ at a}}$

$\Rightarrow \Delta F \text{ at a} = 4Ka^3 \Delta R$

So $\frac{\Delta F \text{ at a}}{\text{F at a}} = \frac{4Ka^3 \Delta R}{Ka^4} = \frac{4\Delta R}{a}$

$= 4\frac{\Delta R}{a} = 4 \times$ relative change in R

11. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 7 - x^2$



want to maximize Area $= w \cdot h$

note $w = 2 \cdot x$

and $h = y = 7 - x^2$

so Area $= 2x(7-x^2) = 14x - 2x^3$

To maximize we need to find the extrema...

Area' $= 14 - 6x^2$

$0 = 14 - 6x^2$

$-14 = -6x^2$

$\frac{-14}{-6} = x^2 \Rightarrow x = \pm\sqrt{7/3}$

Area'(-2) Area'(0) Area'(2)

$- \quad -\sqrt{7/3} \quad + \quad \sqrt{7/3} \quad -$

max

max when $x = \sqrt{7/3}$

so width $= 2\sqrt{7/3}$

and height $= 7 - 7/3 = \frac{14}{3}$

12. A truck has a minimum speed of 9 mph in high gear. When traveling x mph, the truck burns diesel fuel at the rate of

$$0.003935\left(\frac{675}{x} + x\right)\frac{\text{gal}}{\text{mile}}$$

Assume that the truck can not be driven over 63 mph, that diesel fuel costs \$2.84 a gallon, and that the driver is paid \$12 an hour. Find the speed that will minimize the cost of a 500 mile trip.

Total Cost = Cost of Gas + Cost of driver

$= 5.5877\left(\frac{675}{x} + x\right) + \frac{6000}{x}$

$= \frac{3771.6975}{x} + 5.5877x + \frac{6000}{x}$

$= \frac{9771.6975}{x} + 5.5877x$

Total cost'$(x) = -\frac{9771.6975}{x^2} + 5.5877$

Critical Points when Total Cost'$(x) = 0$

$0 = -\frac{9771.6975}{x^2} + 5.5877$

$\Rightarrow 5.5877x^2 = 9771.6975$

$\Rightarrow x = \pm 41.82$ mi/hr

Cost of Gas:

$.003935\left(\frac{675}{x} + x\right)\frac{\cancel{\text{gal}}}{\cancel{\text{mile}}} \cdot 500\,\cancel{\text{mile}} \cdot 2.84\,\frac{\$}{\cancel{\text{gal}}}$

$= 5.5877\left(\frac{675}{x} + x\right)$ dollars

Cost of driver:

$12\,\frac{\$}{\cancel{hr}} \cdot \frac{\cancel{hr}}{x\,\cancel{\text{miles}}} \cdot 500\,\text{miles}$

$= \frac{6000}{x}$ dollars

note -41.82 mi/hr won't work

verify 41.8 mi/hr is a min

$- \quad + $

Cost'(1) 41.8 Cost'(50)

Min total cost @ 42 mi/hr