Exam 1 — TMath 124 — Spring 2012 — Key

1. [6] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F. Let f and g be functions and x and y be positive numbers.

T **F** $-2^2 = 4$. $-2^2 \neq (-2)^2 = (-2)(-2) = 4$ PEMDAS

T F $(x+1)^{\frac{3}{2}} = \left(\sqrt{x+1}\right)^3$.

T F $\dfrac{x^2-1}{x^2} = 1 - x^{-2}$. $\dfrac{x^2-1}{x^2} = \dfrac{x^2}{x^2} - \dfrac{1}{x^2}$

T **F** If $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = 0$, then $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$ does not exist.

ex $f(x) = x(x-1)$
$g(x) = x-1$
$\lim_{x\to 1} \dfrac{f(x)}{g(x)} = \lim_{x\to 1} \dfrac{x\cancel{(x-1)}}{\cancel{(x-1)}}$
$= \lim_{x\to 1} x = 1$

T F $\lim_{x\to a}(g(x)) = 3$, then $\lim_{x\to a} 5 \lim_{x\to a} g(x) = 15$.

T **F** If $\underbrace{\lim_{x\to a} f(x) = f(a)}_{f \text{ is cont}}$, then $f'(a)$ exists.

Show your work for the following problems. The correct answer with no supporting work will receive NO credit (this includes multiple choice questions).

2. [3] (§2.2 #16 & §2.7 #22) Sketch the graph of an example function f that satisfies the following conditions:

function +.5

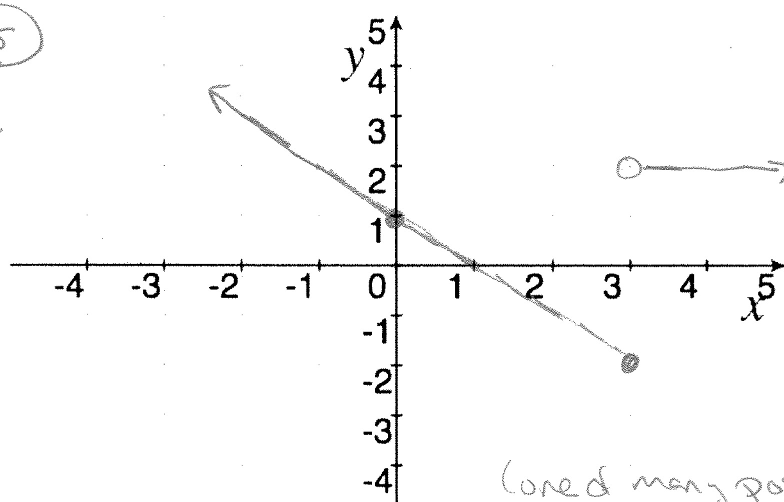
(a) f is continuous everywhere but when $x = 3$ ✓ +.5

(b) $\lim_{x\to 3^+} f(x) = 2$ ✓ +.5

(c) $\lim_{x\to 3} f(x)$ does not exist ✓ +.5

(d) $f(0) = 1$ ✓ +.5

(e) $f'(0) = -1$ ✓ +.5



(one of many possible solutions)

[3] Find a formula for the above graph.

+.5 for each prop above.

$$f(x) = \begin{cases} -x+1 & \text{if } x \leq 3 \\ 2 & \text{if } 3 < x \end{cases}$$

3. Consider the piecewise-defined function f defined below

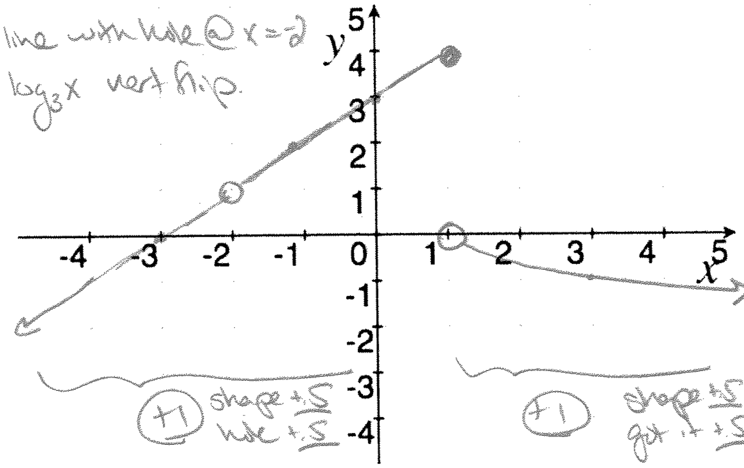
$\frac{(x+2)(x+3)}{x+2}$

$$f(x) = \begin{cases} \frac{x^2+5x+6}{x+2} & \text{if } x \leq 1 \\ -\log_3(x) & \text{if } 1 < x \end{cases}$$

line with hole @ $x=-2$

$\log_3 x$ vert flip.

(a) [3] (Quiz2 #1) Draw the graph on the axis provided.



(+1) shape +.5 hole +.5

(+1) shape +.5 got it +.5

endpts (+1)

(b) Use the graph above to estimate the following (if they exist!):

[1] (WebHW2 #2)
$\lim_{x \to -2} f(x)$

1

[2] (§2.3 #2e)
$\lim_{x \to 3} x^2 f(x)$

$= \lim_{x \to 3} x^2 \cdot \lim_{x \to 3} f(x)$ (+1)

$= 3^2 \cdot (-\log_3 3) = 9(-1) = -9$ (+.5) (+.5)

[1] (WebHW2 #2)
$\lim_{x \to 1^-} f(x)$

4

[1] (limit wks #2)
$\lim_{x \to 1} f(x)$

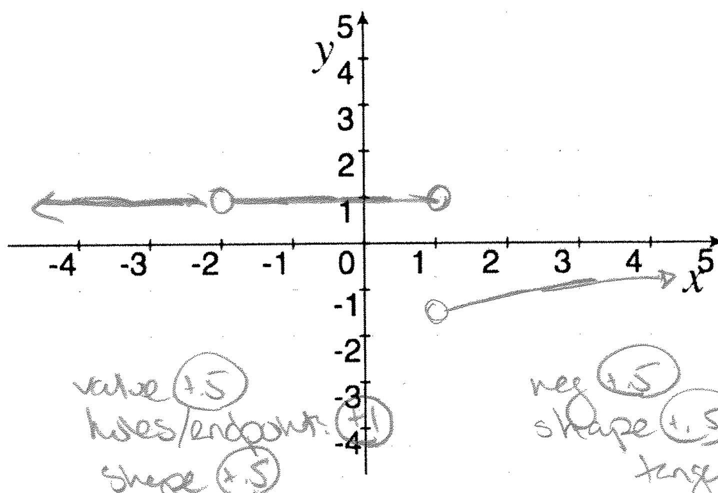
DNE

[1] (WebHW5 #9)
$f'(0)$ = slope of line tangent to f when $x=0$

$= 1$

[1] (WebHW5 #9)
$\frac{d}{dx} f(1)$ DNE

(c) [3] (§2.8 #8) Make a rough sketch of the graph of $f'(x)$:



value (+.5)
holes/endpoints (+1)
shape (+.5)

neg (+.5)
shape (+.5)
tangent lines are becoming more horizontal

4. Find the limit or explain why it does not exist.

[2] (Quiz1 #2)

$$\lim_{x\to 2} \frac{x\ln(x) - 2\ln(x)}{x^2-4}$$

$$= \lim_{x\to 2} \frac{\ln(x)\cancel{(x-2)}}{(x+2)\cancel{(x-2)}}$$

$$= \lim_{x\to 2} \frac{\ln(x)}{x+2}$$

$$= \frac{\ln 2}{4} \approx .173$$

alg (+1)
notation (+.5)
limit (+.5)

[3] (inf limits wks)

$$\lim_{x\to\infty} \frac{x-2}{x^2+1} \quad \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x^2} - \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 - \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x} - \lim_{x\to\infty} \frac{2}{x^2}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{0-0}{1-0} = 0$$

factor (+.5)
right factor (+.5)
notation (+.5)
" $\frac{1}{\text{Big}}$ " = 0 (+.5)
limit (+.5)
alg (+.5)

[3] (§2.3 #39)

$$\lim_{x\to 0} x^6 \cos\left(\frac{3}{x}\right)$$

Recall $-1 \le \cos(3/x) \le 1$ } (+.5)

Since $x^6 \ge 0$ for all x values } (+.5)

we can mult the above inequality thru by x^6

$\rightarrow$ $-x^6 \le x^6\cos(3/x) \le x^6$ } (+.5)

Notice $\lim_{x\to 0} -x^6 = 0 = \lim_{x\to 0} x^6$ } (+.5)

med using (+.5)
used (+.5)

{ Thus by the squeeze/cops & robbers thm

$$\lim_{x\to 0} x^6\cos(3/x) = 0$$

[2] (WebHW4 #4)

$$\lim_{x\to 9} \frac{16+\sqrt{x}}{\sqrt{16+x}}$$

by continuity

$$\frac{16+\sqrt{9}}{\sqrt{16+9}}$$

$$= \frac{16+3}{\sqrt{25}}$$

$$= \frac{19}{5}$$

alg (+1)
notation (+.5)
limit (+.5)

5. [3] (Winter '12 Exam1) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 6$ if $g(6) = -2$ and $g'(6) = 3$.

looking for $y = mx + b$ (+.5)

(+.5) m = slope of line tangent to g when $x = 6$ $= g'(6)$

(+.5) $= 3$

B/c the line is tangent to g at $(6, g(6)) = (6, -2)$ we know (+.5)

$-2 = 3(6) + b \Rightarrow b = -20$ (+.5)

$\Rightarrow y = 3x - 20$

alg/notation (+.5)

6. Let $f(x) = \frac{1}{x} + e^2$.

(a) [2] (§3.1 #17) Find $f'(x)$

alg (+.5)

$$f'(x) = \left(\frac{1}{x} + e^2\right)' = (x^{-1})' + (e^2)' = \underbrace{-1x^{-2}}_{(+.5)} + \underbrace{e^2(x^0)'}_{(+1)}$$

$$= \frac{-1}{x^2} + 0e^2x^{-1} = -1/x^2$$

(b) [4] (§2.8 #34) Find the derivative of f using the definition of derivative. That is, use $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ and verify your answer to part (a).

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{x+h} + e^2\right) - \left(\frac{1}{x} + e^2\right)}{h} \quad (+1)$$

sign (+.5)
fact (+.5)
alg/division (+.5)
notation (+.5)

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} + \cancel{e^2} - 1/x - \cancel{e^2}}{h} = \lim_{h\to 0} \frac{\frac{x}{x}\frac{1}{x+h} - \frac{1}{x}\frac{(x+h)}{(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)} \div h = \lim_{h\to 0} \left[\frac{-h}{x(x+h)} \div \frac{h}{1}\right]$$

$$= \lim_{h\to 0} \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \underbrace{\lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)}}_{(+.5)} = \frac{-1}{x^2}$$

match (a) (+.5)

7. [6] Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

(a) (Story Wks #4) A tank contains C liters of pure water. Brine that contains S grams of salt per liter of water is pumped into the tank at a rate of R liters per minute.

i. [3] Find a formula for the concentration of brine as a function of time t.

ii. [2] Find out what happens to the concentration as $t \to \infty$.

iii. [1] Interpret your answer to part ii in terms your 12 year old niece would understand.

(b) A rock thrown upwards on planet Mars with velocity $15\frac{\text{m}}{\text{s}}$ has a height $h(t) = 15t - 1.86t^2$ meters t seconds later.

i. [2] (Story wks #6) Find the velocity of the rock after 2 seconds.

ii. [1] (Story wks #5) Recall gravity is the constant acceleration experienced by an object from the planet. Find the gravity on Mars.

iii. [3] (Story wks6b) Use calculus to find *when* does the rock reach its maximum height?

a) i) Brine after t seconds $\quad S\left(\frac{\text{grams}}{\cancel{\text{Liter}}}\right)\cdot R\left(\frac{\cancel{\text{Liters}}}{\cancel{\text{min}}}\right)\cdot t(\cancel{\text{min}}) \quad \text{(grams)}$ +1

Total amount of liquid after t seconds

original amount + added amount

$C\,(\text{Liters}) + R\left(\frac{\text{Liters}}{\cancel{\text{min}}}\right)\cdot t(\cancel{\text{min}}) \quad (\text{Liters})$ +1

So the concentration would be

$$\frac{\text{Brine (grams)}}{\text{Total liquid (Liters)}} = \frac{SRt}{C+Rt} \quad +1$$

ii)
$$\underbrace{\lim_{t\to\infty}\frac{SRt}{C+Rt}}_{+.5}\ \underbrace{\frac{1/t}{1/t}}_{+.5} = \lim_{t\to\infty}\frac{\frac{SRt}{t}}{\frac{C}{t}+\frac{Rt}{t}} = \lim_{t\to\infty}\frac{SR}{\frac{C}{t}+R}$$

$$= \frac{\lim_{t\to\infty} SR}{\underbrace{\cancel{\lim_{t\to\infty}\frac{C}{t}}}_{0} + \lim_{t\to\infty} R} = \frac{SR}{R} = S$$

find limit +1

iii) As we add more and more Brine (really salty stuff) to the clean water we had, our total water will start to taste more & more like the salty stuff we added

1.86
2
72

(b) i) Recall velocity $= h'(t)$ } (+.5)

(+1) { So velocity $= v(t) = [15t - 1.86t^2]'$

$= [15t]' - [1.86t^2]' = 15 - 3.72t$

(+.5) { Then the velocity after 2 seconds is

$v(2) = 15 - 3.72(2)$

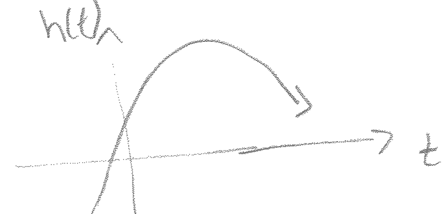
$= 15 - 7.44 = 7.56 \ m/s$

ii) Recall acceleration $= v'(t)$ } (+.5)

(+.5) { So acceleration $= [15 - 3.72t]'$

$= [15]' - [3.72t]' = -3.72$

iii) $h(t)$ looks something like



Notice the highest point is at the vertex of the parabola (so we could do this problem w/ precalc methods)

(+.5) { and also happens to be when the tangent line is horizontal (more of a calculus approach)

(+1) { We want to find x so that

slope of line tangent to h at x = slope of horiz line

ie $h'(x) = 0$ from i we know h' so

alg (+.1)

(+.5) { $\Rightarrow 15 - 3.72t = 0$

$\Rightarrow -3.72t = -15$

$\Rightarrow t = \frac{+15}{+3.72}$ ★

is the time rock reaches max