EXAM 1 | TMath 124 | Autumn 2018

Key

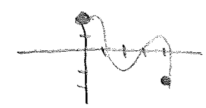
1. [8] TRUE/FALSE: Let f and g be functions. Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

T **F** $x + \frac{1}{x} = \frac{x+1}{x}$ $\qquad \frac{x}{x}\frac{x}{1} + \frac{1}{x} = \frac{x^2+1}{x}$

T **F** $\frac{1}{x^2} = x^{\frac{1}{2}}$ $\qquad \frac{1}{x^2} = x^{-2} \qquad x^{1/2} = \sqrt{x}$

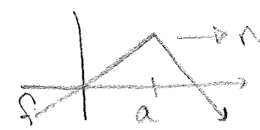
T **F** If $\lim_{h\to 0} g(h) = 0$ then $\lim_{h\to 0} \frac{f(h)}{g(h)}$ does not exist. ex $\lim_{h\to 0} \frac{(1+h)^2 - 1^2}{h}$ exists (ended)

T F If f is a continuous function, $f(0) = 2$, and $f(4) = -2$, then the graph of f has an x-intercept between 0 and 4.



by Intermediate Value Thm

T **F** If f is continuous at a, then f is differentiable at a.



T **F** $(e^x)' = xe^{x-1}$. $\qquad (e^x)' = e^x$

T F $\frac{d}{dx}(\sin(x)) = \cos(x)$.

T **F** $x^3 = 3x^2$ $\qquad (x^3)' = 3x^2$

↳ need the derivative?

Show *all* your work (numerically, algebraically, or geometrically) for the following problems. Supporting work is needed to earn credit.

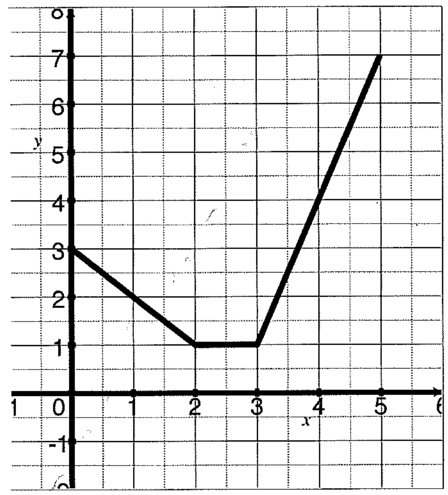
2. [4] Let f be a function. Explain what $f'(2)$ is to a fifth grader. start (+.5)

$f'(2)$ is essentially how steep (+1) the graph of f (+.5) is above 2 (+1)

sense (+.5) -

Correct / may not be complete (+.5)

3. The following graph represents the distance d (in inches) an ant is from Dr. Vanderpool after x seconds.



(a) [1] How far is the ant from Dr. Vanderpool initially?

3 inches

(b) [6] Estimate the following, if possible:
(9/27Lecurer, LimitActivity #2, WebHW2 #5, PracticeExam#3, DerivativeActivity#1)

$d(2)$

1 inch (+1)

$\lim_{x\to 1} d(x)$

2 (+1)

$\lim_{x\to 0^-} d(x)$

DNE (+1) nse 3 +.5

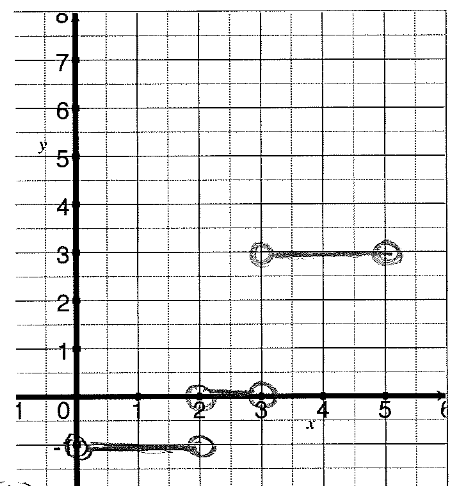
$\lim_{x\to 3} (2d(x) - 6)$ $= 2\lim_{x\to 3} d(x) - 6$ (+.5)

$= 2\cdot 1 - 6 = -4$ (+.5) (+.5)

$\frac{d}{dx} d|_{x=4}$

= slope of line tang to d @ x=4 (+1)

$= \frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$ (+.5)



4. [3] (§3.2 #104) On the blank axis sketch the graph of the ant's velocity.

note: the ant must have noticed Dr Vanderpool @ 3 sec & started running away.

values (+1.5)
shape (+1)
endpoints (+.5)

5. [6] Find the limit if it exists, or explain why it does not exist.

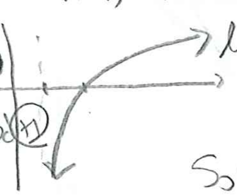
(§2.4 #114)

$$\lim_{x\to 1^+} \ln(x-1)$$

~~$\lim_{x\to 1^+} \ln(x-1) = \ln(1-1) = \ln(0)$~~

$\ln(x-1)$ looks like $\ln x$ shifted right 1

shape (+.5)
shift graph (+.5)
limit one sided (+1)
got it (+1)
notation (+.5)



as $x \to 1^+$

$y \to -\infty$

So $\lim_{x\to 1^+} \ln(x-1) = -\infty$

or

Table (+.5)

x	$\ln(x-1)$
1.01	
1.0001	
1.000001	

correct eval (+.5)
values from above (+1)
got it (+.5)
notation (+.5)
reasoning (+.5)

(LimitLawActivity #1)

$$\lim_{x\to\pi} \frac{2}{x-3} + \cos(x)$$

notation (+.5)

(+1)

$\frac{2}{\pi-3} + \cos(\pi) = \frac{2}{\pi-3} - 1$ ≈ 13.29

plug in (+1)

or

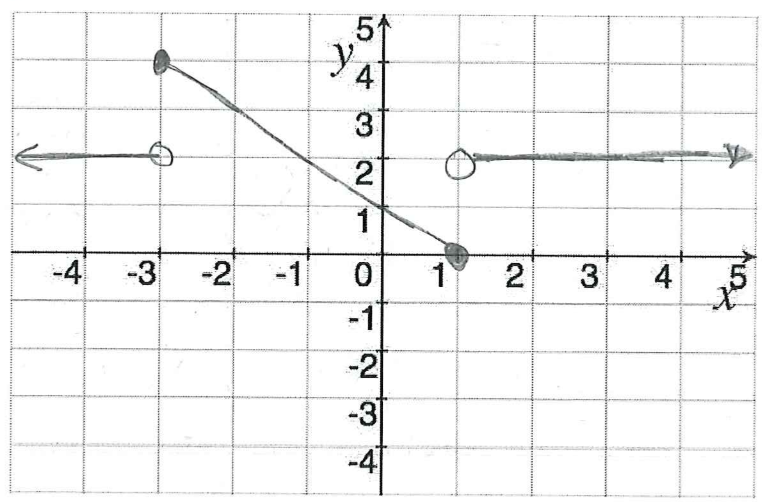
6. [5] (Quiz2 #2) Draw a graph for a function $\alpha(x)$, that satisfies all of the following:

(a) $\lim_{x\to\infty} \alpha(x) = 2$, (+1)

(b) α is not continuous at $x = -3$, (+1)

(c) $\alpha(-3) = 4$ (+1)

(d) $\alpha'(0)$ is negative, (+1)



function (+1)

7. [3] (2016Exam1 #2) If the tangent line to $y = f(x)$ at $(1, 0)$ passes through the point $(-1, -3)$. Find the following:

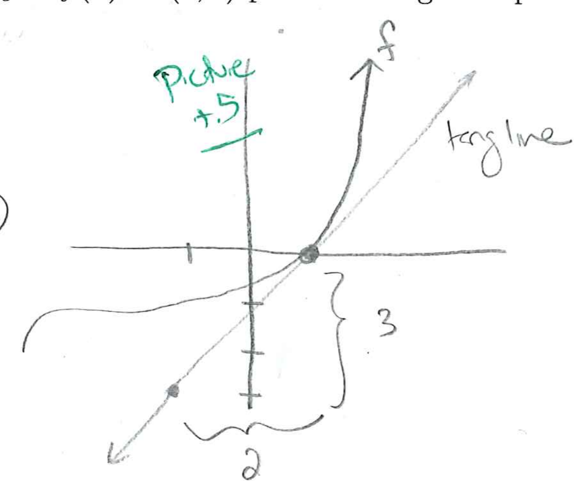
(a) $f(1)$ $= 0$ (+1)

(b) $f'(1)$ $=$ slope of line tang to f (+1)

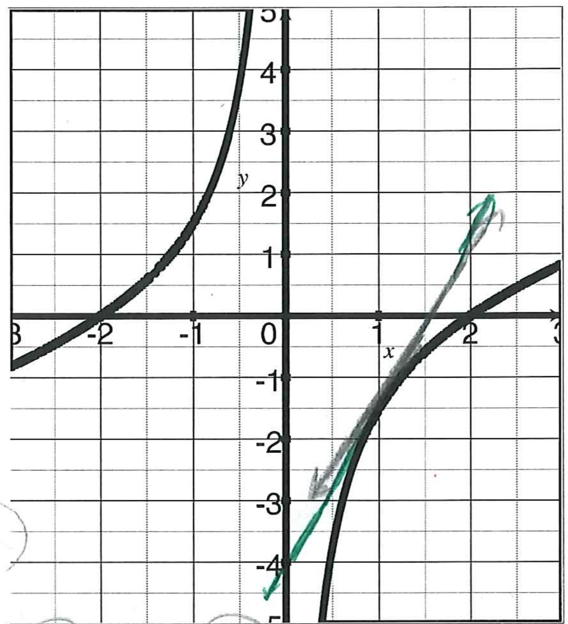
$= \frac{\Delta y}{\Delta x}$

$= \frac{3}{2}$ (+1)

Picture +.5



8. Consider $g(x) = \dfrac{\frac{1}{2}x^2 - 2}{x}$ graphed to the right.



(a) [4] (WebHW6 #7) Find $\dfrac{dg}{dx}$

(+2) $g(x) = \dfrac{\frac{1}{2}x^2}{x} - \dfrac{2}{x} = \frac{1}{2}x - 2x^{-1}$

(+.5) Power Rule $\Rightarrow dg/dx = \frac{1}{2} - 2(-1)x^{-2}$

$dg/dx = \frac{1}{2} + 2x^{-2}$ or $\frac{1}{2} + \frac{2}{x^2}$ or $\frac{x^2+4}{2x^2}$

(+1.5)

or

$dg/dx = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h}$ (+.5) $= \lim\limits_{h\to 0} \dfrac{\frac{\frac{1}{2}(x+h)^2-2}{x+h} - \frac{\frac{1}{2}x^2-2}{x}}{h}$ (+.5) (+.5) $= \lim\limits_{h\to 0} \left[\dfrac{\frac{1}{2}(x^2+2xh+h^2)-2}{x+h} - \dfrac{\frac{1}{2}x^2-2}{x}\right] \div h$

$= \lim\limits_{h\to 0} \left[\dfrac{\frac{1}{2}x^2+xh+\frac{1}{2}h^2-2}{x+h} - \dfrac{\frac{1}{2}x^2-2}{x}\right] \div \frac{h}{1} = \lim\limits_{h\to 0} \dfrac{\frac{1}{2}x^3+x^2h+\frac{1}{2}xh^2-2x-(\frac{1}{2}x^2-2)(x+h)}{(x+h)x} \cdot \frac{1}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{\frac{1}{2}x^3}+x^2h+\frac{1}{2}xh^2-\cancel{2x}-\cancel{\frac{1}{2}x^3}-\frac{1}{2}x^2h+\cancel{2x}+2h}{(x+h)xh} = \lim\limits_{h\to 0} \dfrac{\cancel{h}(x^2+\frac{1}{2}xh-\frac{1}{2}x^2+2)}{(x+h)x\cancel{h}}$

$= \dfrac{x^2+\frac{1}{2}x(0)-\frac{1}{2}x^2+2}{(x+0)x} = \dfrac{\frac{1}{2}x^2+2}{x^2} = \frac{1}{2}+2x^{-2}$

alg/fractions (+1.5)
eval limits (+1)

(b) [2] (Quiz2 #3) Find $g'(1)$

$g'(1) = \frac{1}{2} + 2(1)^{-2} = \frac{1}{2} + 2 = \frac{5}{2}$ or 2.5

plug 1 into (a) (+1)

got it (+.5) alg (+.5)

(c) [3] (Poly&ExpActivity #4) Find the equation of the line tangent to g when $x = 1$.

(+.5) Looking for $y = mx + b$ or $y - y_1 = m(x - x_1)$

m = slope of line tangent to g @ $x = 1$

$= g'(1)$ (+.5)

$= 5/2$ from (b)

thru (+.5) $(1, g(1)) = \left(1, \dfrac{\frac{1}{2}(1)^2-2}{1}\right) = (1, -3/2)$ ~~(.5)~~

So $y - {-3/2} = 5/2(x-1)$

or $y + 3/2 = 5/2x - 5/2$

or $y = 5/2x - 4$

plug in (+1)

Fun Fact: Everyone did b.

9. Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
   No, doing both questions will not earn you extra credit.

   (a) (2013WinterExam1) Under certain assumptions the velocity $v(t)$ of a falling raindrop at time t is:
   $$v(t) = v^*(1 - e^{\frac{-gt}{v^*}})$$
   where g is the acceleration due to gravity (9.8 m/s^2).

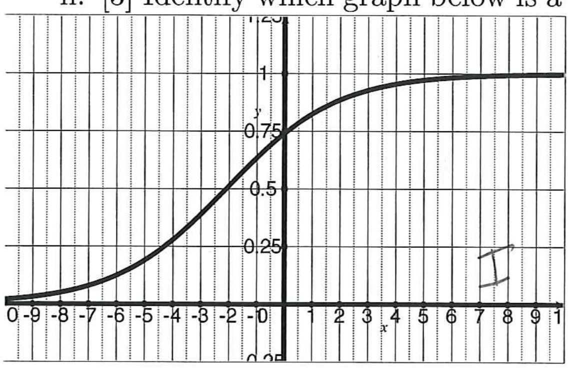
   i. [3] Find $\lim_{t\to\infty} v(t)$.

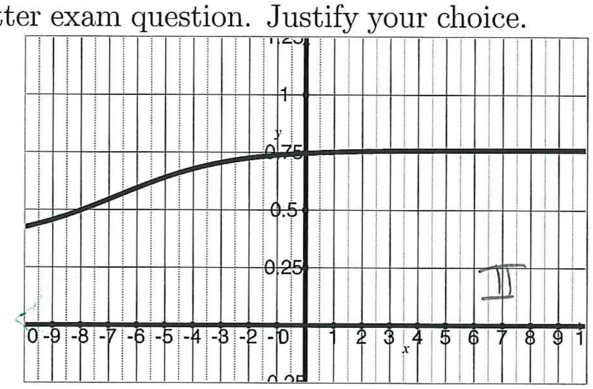
   ii. [2] Interpret the answer given in (i) as a scientist and explain what v^* is in everyday language.

   (b) (WordProblems #1) Test makers use item response functions $P(x)$ to determine the difficulty and effectiveness of a given test question. The variable x is the ability of a test taker and $P(x)$ is the probability that the test taker gets the problem correct. By convention we let an "average ability" correspond with $x = 0$. Thus $P(0) = .75$ means that a person with average ability has a 75% chance of getting the question correct.

   i. [2] Assume the question is multiple choice with 3 ~~answers~~ choices, find $\lim_{x\to-\infty} P(x)$. Justify yourself.

   ii. [3] Identify which graph below is a better exam question. Justify your choice.





(a) start (+.5)

i) $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} v^*(1-e^{-\frac{gt}{v^*}})$ (+.5)

$= v^* \lim_{t\to\infty}\left(1 - \frac{1}{e^{gt/v^*}}\right) = v^*\left(1 - \cancel{\lim_{t\to\infty}\frac{1}{e^{gt/v^*}}}^{\,0}\right)$

$= v^*(1-0) = v^*$

limit laws (+1)
limit of e^{-gt/v^*} (+1)

ii) v^* is the velocity a raindrop approaches & doesn't exceed. (+1)
Often it is called the terminal velocity (when friction & acceleration balance) (+1)

(b) start (+.5)   notation (+.5)

i) $\lim_{x\to-\infty} P(x) = \frac{1}{3}$ (+.5)

(+1) As x goes to $-\infty$, their abilities go down & down. Their choices are likely to become more random meaning they have a 1 in 3 chance of getting the problem correct

ii) (+.5) graph read (+1) interpret

I is a better choice (+1)
Generally we want people with higher abilities to be able to get the problem correct (probabilities ≈ 1). The question on the right has prop. of high ability people w/ probabilities < 1