Practice TQS 120 Final

NAME: This is a sample final to be used for practice. This is *not a template* for the Final that will be given in class. Many of the questions on the Final will look quite di?erent than those appearing here.

[10] Let f & g, be functions.

T F $(f \circ g)(x) = (g \circ f)(x)$ ex $f(x) = x+1$ $g(x) = 3x$

T F $(\frac{f}{g})(x) = (\frac{g}{f})(x)$ $(f \circ g)(x) = 3x+1$ but $(g \circ f)(x) = 3(x+1)$

T F $\sqrt{(x^2)} = x$ for all real numbers x. Let x be -2

~~T F If $h(x) = x^2 + 1$, then h is an even function~~. Should not be on this test.

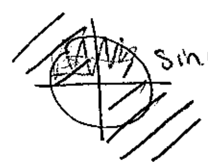
T F $\ln \frac{x}{y} = \ln x - \ln y$ for all positive numbers x and y.

T F $\log(\log(10)) = 0$. $\log(1) = 0$ ✓

T F ~~Just as every integer is either even or odd, every function is either an even function or odd function~~. should not be on this test.

T F $\sin(\frac{\pi}{3} + x) = \sin \frac{\pi}{3} + x$

T F If $\sin\theta > 0$ and $\tan\theta < 0$, then $\cos\theta < 0$



$\sin\theta > 0$

T F The range of $\sin^{-1}$ is $[0, \pi]$

range is from $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Right answers will *not* get credit without supporting work. Note "undefined" and "no solution" are possible answers.

X

1. Find all x such that

$$2(5 + (8 - x)^2)^{-\frac{1}{2}} - 1 = 0$$

$$2 \frac{1}{\sqrt{5+(8-x)^2}} - 1 = 0$$

$$\frac{2}{\sqrt{5+(8-x)^2}} = 1$$

$$2 = \sqrt{5+(8-x)^2}$$

$$4 = 5 + (8-x)^2$$

$$-1 = (8-x)^2$$

$$\pm\sqrt{-1} = 8 - x$$ does not exist any real solutions.

[I should have choosen a problem more like

$$2(5-(8-x)^2)^{-\frac{1}{2}} - 1 = 0$$

for a more interesting answer.]

2. [2] Explain what a function is.

A function is two sets (a domain & a range) & a rule between them such that every number in the domain is sent to exactly one output.

3. Given $m(x) = \dfrac{x}{x-5}$, and $n(x) = \sqrt{4x-8}$,

(a) [4] If $p(x) = 3m(x+1)$, find the domain and rule of p.

$$3\left[\frac{(x+1)}{(x+1)-5}\right]$$

$$= \frac{3x+3}{x-4}$$

note: I was not told to simplify

domain: all #'s so that denominator $\neq 0$

$\Rightarrow \quad x-4 \neq 0$

$x \neq 4$

or $(-\infty, 4) \cup (4, \infty)$

(b) [3] Find the domain and rule of $n \circ m$.

$$(n \circ m)(x) = n(m(x))$$

$$= n\left(\frac{x}{x-5}\right)$$

$$= \sqrt{4\left(\frac{x}{x-5}\right) - 8}$$

domain: all #'s so that denominator $\neq 0$ AND stuff under sqrt ≥ 0

$x-5 \neq 0$

$\Rightarrow x \neq 5$

$4\left(\frac{x}{x-5}\right) - 8 \geq 0$

$\frac{4x}{x-5} \geq 8$

consider $\frac{4x}{x-5} = 8$

$4x = 8x - 40$

$+4x = +40$

$x = 10$

smaller than 8 — 10 — larger than 8

$\Rightarrow \quad x \geq 10$

So $[10, \infty)$

(c) [5] Find the domain and rule of $\dfrac{n}{m}$.

$$\left(\frac{n}{m}\right)(x) = \frac{n(x)}{m(x)}$$

$$= \frac{\sqrt{4x-8}}{\frac{x}{x-5}}$$

domain: all #'s so that den of den $\neq 0$ AND den $\neq 0$

$x-5 \neq 0$

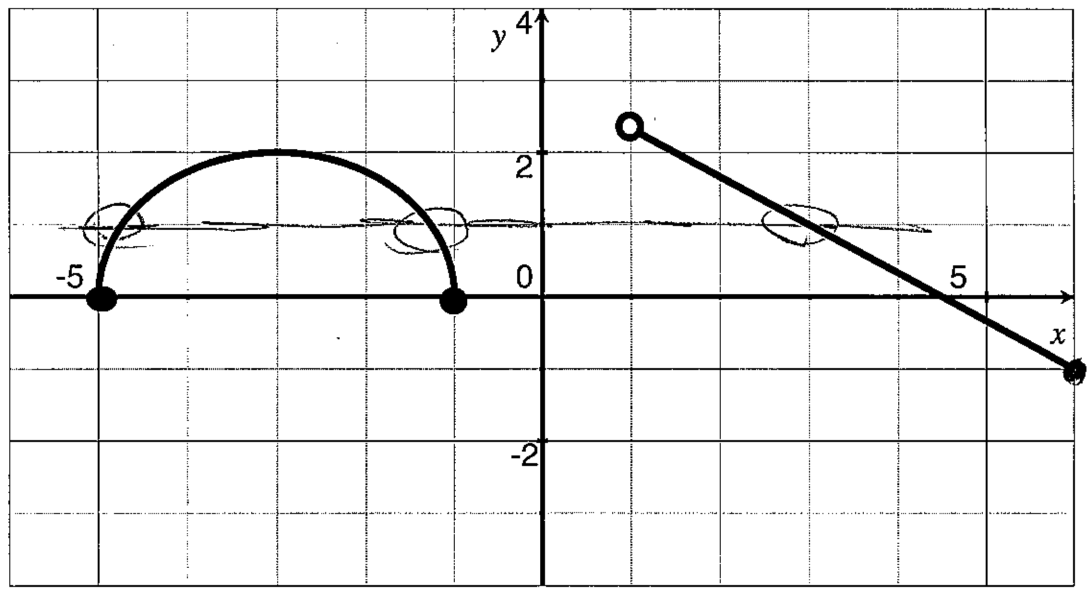
$x \neq 5$

$\frac{x}{x-5} \neq 0$

$x \neq 0$

So $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$

4. [3] Let the following be the graph of g.



(a) What is the domain of g?

$[-5, -1] \cup (1, 6)$

(b) The function g is a piecewise defined function consisting of a straight line and a semicircle. Write down the rule for g.

Semicircle w/ center at $(-3, 0)$ and radius 2

$(x - -3)^2 + (y-0)^2 = 2^2$

$(x+3)^2 + y^2 = 4$

$y = \sqrt{4-(x+3)^2}$

line w/ slope $-\frac{2}{3}$ + through $(3,1)$

$1 = -\frac{2}{3}(3) + b$

$\leadsto b = 3$

$y = -\frac{2}{3}x + 3$

So

$g(x) = \begin{cases} \sqrt{4-(x+3)^2} & \text{if } -5 \leq x \leq \\ -\frac{2}{3}x + 3 & \text{if } 1 < x \leq \end{cases}$

(c) Use the graph above to *estimate* all x value(s) so that $g(x) = 1$?

Circled above

$x \approx -4.7, -1.3$ and 3

this can be found exactly by solving for x in $1 = \sqrt{4-(x+3)^2}$ and $1 = -\frac{2}{3}x + 3$

(d) Find the total length (of the curve and the line) that is graphed above.

Total length = length of semicircle + length of line

$= 2\pi + \sqrt{25 + \frac{16}{9}}$

semicircle:

$\frac{1}{2}[2\pi \cdot 2] = 2\pi$

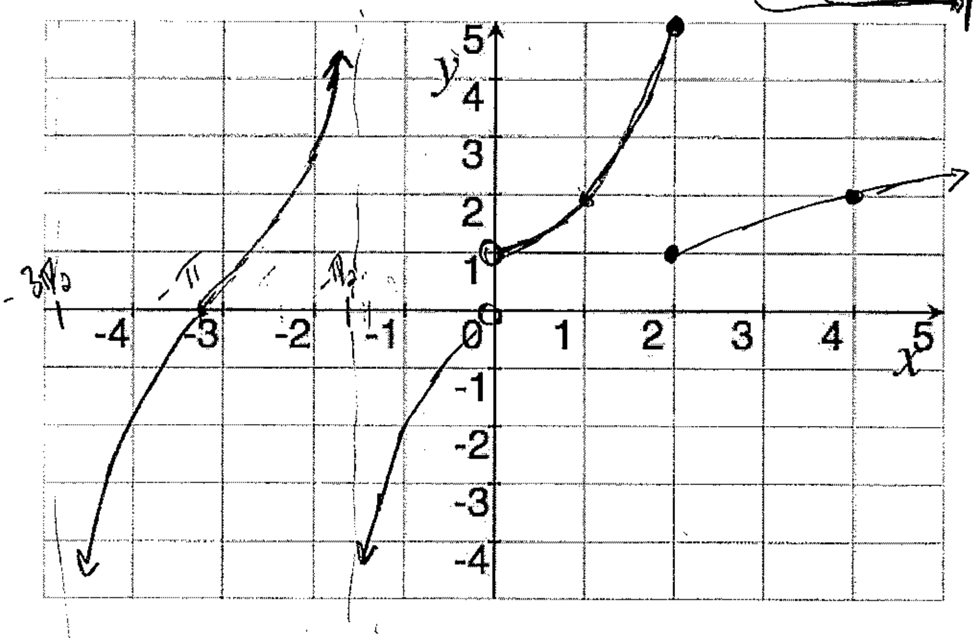
line: connects $(6,1)$ to $(1, \frac{7}{3})$

$\sqrt{5^2 + (\frac{4}{3})^2}$

5. Define f by

$$f(x) = \begin{cases} \tan x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 < x \leq 2 \\ \log_2(x) & \text{if } x \geq 2 \end{cases}$$

vert shift up one



note: this is not a function b/c it fails the vertical line test at x = 2

(a) [8] Graph f on the axes above.

(b) [9] Find the following if possible:

$f(1)$

2

$f(2) + f(3)$

not defined b/c

$f(2) = 5$ AND 1

$f(0)$

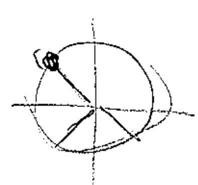
not defined

$f(\frac{-13\pi}{4})$ $= \tan(-13\pi/4)$

$= \tan(-8\pi/4 - 5\pi/4)$

$= \tan(-2\pi - 5\pi/4)$

$= \tan(-5\pi/4)$ b/c periodic



$= \dfrac{\sin -5\pi/4}{\cos -5\pi/4} = \dfrac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$

Range of f

$\mathbb{R}$

6. [16] Find all values x that satisfy the following (be sure to check your answers):

$|-2x-6| = 2$

$-2x-6=2$ or $-2x-6=-2$

$-2x=8$ $\quad$ $-2x=4$

$x=-4$ $\quad$ $x=-2$

Check:

$|-2(-4)-6| = 2$ ✓ $\quad$ $|-2(-2)-6| = 2$ ✓

$5^{4x-1} = 7^x$

$\ln 5^{4x-1} = \ln 7^x$

$(4x-1)\ln 5 = x \ln 7$

$x 4\ln 5 - \ln 5 = x\ln 7$

$x 4\ln 5 - x\ln 7 = \ln 5$

$x(4\ln 5 - \ln 7) = \ln 5$

$x = \dfrac{\ln 5}{4\ln 5 - \ln 7}$

note: you could have used decimal here

7. [6] Assuming that $\log_3 x = 5.3$ and $\log_3 y = 2.1$ find the following exactly:

$\log_3 \dfrac{27x^3}{y^2} = \log_3(27x^3) - \log y^2$

$= \log_3 27 + \log_3 x^3 - \log y^2$

$= \log_3 3^3 + 3\log_3 x - 2\log y$

$= 3 + 3\cdot 5.3 - 2\cdot 2.1$

$= 3 + 26.5 - 4.2 = 25.3$

$5.3 \times 5 = 26.5$ $\quad$ $2.1 \times 2 = 4.2$

$29.5 - 4.2 = 25.3$

$\log_9 3x$

$= \log_9 3 + \log_9 x$

$= \log_9 9^{1/2} + \log_9 x$

$= 1/2 + \log_9 x$

note $\log_3 x = 5.3 \Rightarrow x = 3^{5.3}$

$= 1/2 + \log_9 3^{5.3} = 1/2 + \log_9 (9^{1/2})^{5.3}$

$= 1/2 + \log_9 9^{2.65}$

$= .5 + 2.65$

$= 3.15$

8. [4] Find all exact values for x that satisfy the following:

$\log(x-16) = 2 - \log(x-1)$

$\log(x-16) + \log(x-1) = 2$

$\log(x-16)(x-1) = 2$

$(x-16)(x-1) = 10^2$

$x^2 - 17x + 16 = 100$

$x^2 - 17x - 84 = 0$

$(x-21)(x+4) = 0$

$x = 21$ or $\cancel{-4}$

b/c domain problems.

$3^{5x} 9^x = 27$

$3^{5x}(3^2)^x = 3^3$

$3^{5x} 3^{2x} = 3^3$

$3^{5x+2x} = 3^3$

$3^{7x} = 3^3 \Rightarrow 7x = 3$

$\Rightarrow x = 3/7$

★

9. Simplify: [I should add the condition $c, d > 0$]

$\dfrac{\sqrt{c^2d^6}}{\sqrt{4c^3d^{-4}}} = \dfrac{(c^2d^6)^{1/2}}{(4c^3d^{-4})^{1/2}}$

$= \dfrac{(c^2)^{1/2}(d^6)^{1/2}}{4^{1/2}(c^3)^{1/2}(d^{-4})^{1/2}} = \dfrac{|c|\,|d|^3}{2\, c^{3/2}\, d^{-2}}$

$= \dfrac{1}{2} |c|^{1-3/2} d^{3-2} = \frac{1}{2} c^{-1/2} d^5$

$= \frac{1}{2} |c|^{-1/2}|d|^5$

$2 - \log_5(25z)$

$2 - [\log_5 25 + \log_5 z]$

$2 - \log_5 5^2 - \log_5 z$

$2 - 2 - \log_5 z$

$\log_5 z$

10. [7] Given $f(3) = 0$, use the factor theorem to find the other roots of $f(x)$ $x^4 - 3x^3 - 25x^2 + 75x$

Since $f(3)=0$ 3 is a root

$\Rightarrow$ $x-3$ is a factor of $f(x)$

So
$$\begin{array}{r|l} & x^3 - 25x \quad R0 \\ x-3 & x^4 - 3x^3 - 25x^2 + 75x \\ & -(x^4 - 3x^3) \\ & -(-25x^2 + 75x) \\ & 0 \end{array}$$

So $\dfrac{x^4 - 3x^3 - 25x^2 + 75x}{x-3} = x^3 - 25x$

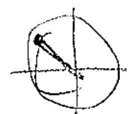
$\Rightarrow x^4 - 3x^3 - 25x^2 + 75x = (x^3 - 25x)(x-3)$

$= x(x^2-25)(x-3)$

$= x(x+5)(x-5)(x-3)$

So the other roots are:
$0, -5, 5 \;\&\; 3$

11. Simplify:



$\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = ?$

$\Rightarrow \sin ? = 1/\sqrt{2}$

$\Rightarrow ? = \pi/4$

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}$$

$$\frac{\cos^2 x + (1-\sin x)^2}{\cos x(1-\sin x)}$$

$$\frac{\boxed{\cos^2 x + 1 - 2\sin x + \sin^2 x}}{\cos x(1-\sin x)} = \frac{1+1-2\sin x}{\cos x(1-\sin x)}$$

$$= \frac{2 - 2\sin x}{\cos x(1-\sin x)}$$

$$= \frac{2\cancel{(1-\sin x)}}{\cos x\cancel{(1-\sin x)}}$$

$$= 2/\cos x$$

12. [4] Let $\frac{-\pi}{2} < \theta < 0$ and $\cos\theta = \frac{1}{5}$. Find $\tan\theta$.

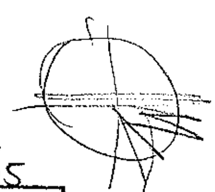
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + (1/5)^2 = 1$

$\sin^2\theta = 1 - 1/25 = 24/25$

$\sin\theta = \pm\sqrt{24/25}$



$\Rightarrow \tan\theta = \dfrac{\frac{-\sqrt{24}}{5}}{1/5}$

$= -\sqrt{24}$

13. [6] Let $\frac{\pi}{2} < \phi < \pi$ and $\frac{-\pi}{2} < \theta < 0$. Given that $\sin\phi = \frac{2}{3}$ and that $\cos\theta = \frac{1}{5}$, find $\cos(\theta + \phi)$. (You are free to use results from #10 above.)

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$= 1/5 \cos\phi - \sin\theta \cdot 2/3$

from #12 $\sin\theta = -\dfrac{\sqrt{24}}{5}$

and to find $\cos\phi$ we use pyth --

$\cos^2\phi + \sin^2\phi = 1$

$\cos^2\theta = 1 - (2/3)^2 = 5/9 \Rightarrow \cos\theta = \pm\dfrac{\sqrt{5}}{3}$

$= 1/5 \cdot \dfrac{-\sqrt{5}}{3} - \dfrac{-\sqrt{24}}{5} \cdot 2/3$

$= \dfrac{-\sqrt{5}}{15} + \dfrac{2\sqrt{24}}{15}$

$= \dfrac{2\sqrt{24} - \sqrt{5}}{15}$

12. [5] Your given a 16 oz mocha that is a rather weak 3% espresso. You, knowing you'll be up late studying mathematics, would rather like a 30% espresso drink. Realizing this you purchase an espresso machine. How much weak mocha do you discard and replace with straight espresso to have a 16 oz mocha with the desired concentration?

let x be the mocha you keep
y be the amount of espresso

$16 \times 3 = 48$

total mocha $\Rightarrow 16 = x + y$

total espresso $\Rightarrow .3 \cdot 16 = .03x + y$

2 equations, 2 unknowns

$16 = x + y$ and $4.8 = .03x + y$

$\Rightarrow 16 - y = x$ sub into

to get $4.8 = .03(16 - y) + y$

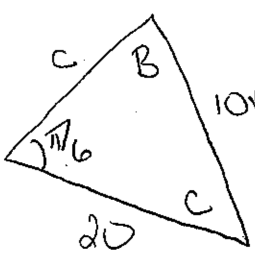
$4.8 = .48 - .03y + y$

$= .48 - .48$

$4.32 = .97y$

$\Rightarrow y = \frac{4.32}{.97} = \frac{432}{97}$ oz

13. [5] Use the conventions from the book and class and let A be measure of the angle opposite the side with length a. Given that $a = 10\sqrt{2}$, $b = 20$, and $A = \frac{\pi}{6}$ with the standard notation, determine if the information describes 0, 1, or 2 triangles and solve for them/it if they/it exist/s.



$\frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin B}{20}$

$\frac{20 \sin \pi/6}{10\sqrt{2}} = \sin B$

$\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \frac{1}{2} = \sin B$

$\Rightarrow B = \pi/4$ or $3\pi/4$

Recall
$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

$\boxed{\text{if } B = \pi/4}$

$C = \pi - \pi/4 - \pi/6$

$= \frac{12\pi - 3\pi - 2\pi}{12}$

$\boxed{\therefore C = 7\pi/12}$

$\Rightarrow \frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin \frac{7\pi}{12}}{c}$

$\frac{1}{20\sqrt{2}} = \frac{\sin \frac{7\pi}{12}}{c}$

$c = 20\sqrt{2} \sin \frac{7\pi}{12}$

$c = 20\sqrt{2} \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$c = 20\sqrt{2}\left[\sin \pi/4 \cos \pi/3 + \cos \pi/4 \sin \pi/3\right]$

$\boxed{c = 20\sqrt{2}\left[\frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}\right] = 10(1+\sqrt{3})}$

$\boxed{\text{if } B = 3\pi/4}$

$C = \pi - 3\pi/4 - \pi/6$

$= \frac{12\pi - 9\pi - 2\pi}{12}$

$\boxed{\therefore C = \pi/12}$

$\frac{\sin \pi/6}{10\sqrt{2}} = \frac{\sin \pi/12}{c}$

$\frac{1}{20\sqrt{2}} = \frac{\sin \pi/12}{c}$

$c = 20\sqrt{2} \sin \pi/12$

$c = 20\sqrt{2} \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$

$c = 20\sqrt{2}\left[\sin \pi/3 \cos \pi/4 + \cos \pi/3 \sin\right.$

$\boxed{c = 10(\sqrt{3} - 1)}$

14. [5] Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope

use $P_0 2^{-t/h} = P(t)$

start with P_0 + end with $\frac{4}{5}P_0$ when $t=3$.

$$\frac{\frac{4}{5}\cancel{P_0}}{\cancel{P_0}} = \frac{\cancel{P_0}\, 2^{-3/h}}{\cancel{P_0}} \quad \text{solve for } h.$$

$$\frac{4}{5} = 2^{-3/h}$$

$$\ln \frac{4}{5} = \frac{-3}{h} \ln 2$$

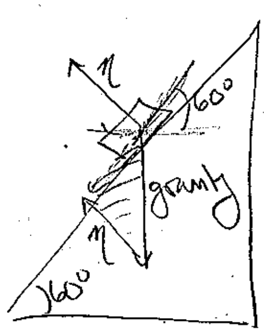
$$\frac{\ln \frac{4}{5}}{\ln 2} = \frac{-3}{h}$$

$$h \frac{\ln \frac{4}{5}}{\ln 2} = -3$$

$$\Rightarrow h = \frac{-3 \ln 2}{\ln \frac{4}{5}}$$

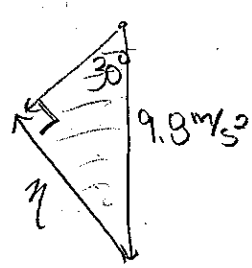
15. [5] The force of friction is sometimes calculated by multiplying the normal force (the force holding the object up) by the mass of the object and by a 'coefficient of friction'. The coefficient of friction is a dimensionless number that depends on the two surfaces being pressed together.

A 10kg block is sliding down a dry glass ramp with angle of elevation of 60° and with a coefficient of friction of .94. Find the force of friction acting on the block.



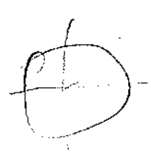
need to find η + then compute $.94\eta$ (mass of obj)

ie $.94\eta \cdot 10\text{kg}$



Sohcahtoa

$$\sin 30° = \frac{\eta}{9.8}$$



$$\Rightarrow \eta = 9.8 \sin 30° = 9.8 \cdot \tfrac{1}{2}$$

So the force of friction is $.94 \cdot 9.8 \cdot .5 \cdot 10 \ \text{m/s}^2 \cdot \text{kg}$