Key

Exam 2 | TMath 120 | Autumn 2021

median 64%
median exam ave 67.5%
overall marks 75%

1. [5] TRUE/FALSE: Circle T in each of the following cases if the statement is *always* true. Otherwise, circle F.

T F $x^2 + (y-2)^2 = 9$ defines a circle with radius 9. $(x-h)^2+(y-k)^2=r^2 \Rightarrow 9=r^2 \Rightarrow r=\sqrt{9}$

T F $\log(x+y) = \log(x) + \log(y)$ for all $x, y > 0$. $\log(x)+\log(y)=\log(x\cdot y)$

T F $\dfrac{\log(x)}{\log(y)} = \dfrac{x}{y}$ for all $x, y > 0$ $\quad \dfrac{\log(x)}{\log(y)} = \log_y(x)$

T F $x^5x^2 = x^{10}$ $\quad (xxxxx)(xx)=x^7$

T F $x^{-2} = \sqrt{x}$ $\quad x^{-2}=\dfrac{1}{x^2} \quad \sqrt{x}=x^{1/2}$

Show all your work. Reasonable supporting work must be shown to earn credit.

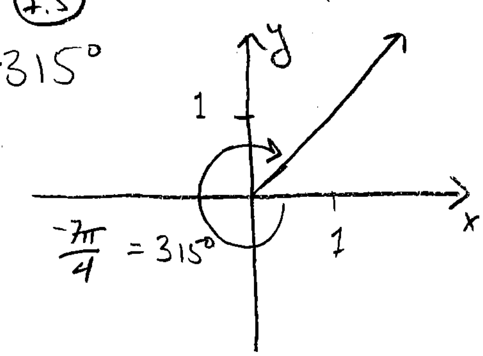
2. [2] (Suggested§4.1 #39) Convert $\frac{-7\pi}{4}$ from radians to degrees. (+.5)

$$\frac{-7\cancel{\pi}\ \cancel{\text{rad}}}{4} \cdot \frac{180°}{\cancel{\pi}\ \cancel{\text{rad}}} = \frac{-7}{4}\cdot 180° = -315°$$

(+.5) conversion factor used correctly (+1)

3. [2] (WebHW§4.1 #6) Draw the angle $\frac{-7\pi}{4}$ radians.

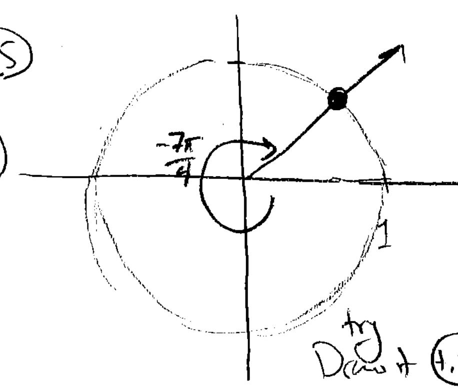
(+.5) direction
(+.5) 2 sides
(+1) correct angle



4. [3] (Circle&AngleActivity #4) Find the point(s) that are both on the graph of the unit circle and the angle $\frac{-7\pi}{4}$ radians.

note the terminal side of $-7\pi/4$ is $x=y$ (+.5)

Must be on the circle: $x^2+y^2=1$ (+.5)



try Draw it (+.5)

So $x^2+x^2=1$ sub (+.5)

$2x^2=1$ algebra (+1)

$x^2=\frac{1}{2}$

$x=\pm\sqrt{\frac{1}{2}}$ $\Rightarrow$ point $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ or $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

12

5. We know that $\log(x) = -2.5$, $\log(z^2) = 6$.

(a) [2] (§3.3 #16) Find z

(+.5) $\log(z^2) = 6$

$\Rightarrow$ (+.5) $10^6 = z^2$

$\sqrt{10^6} = \sqrt{z^2}$

$10^3 = z$

$1000 = z$

Solve for z (+.5)

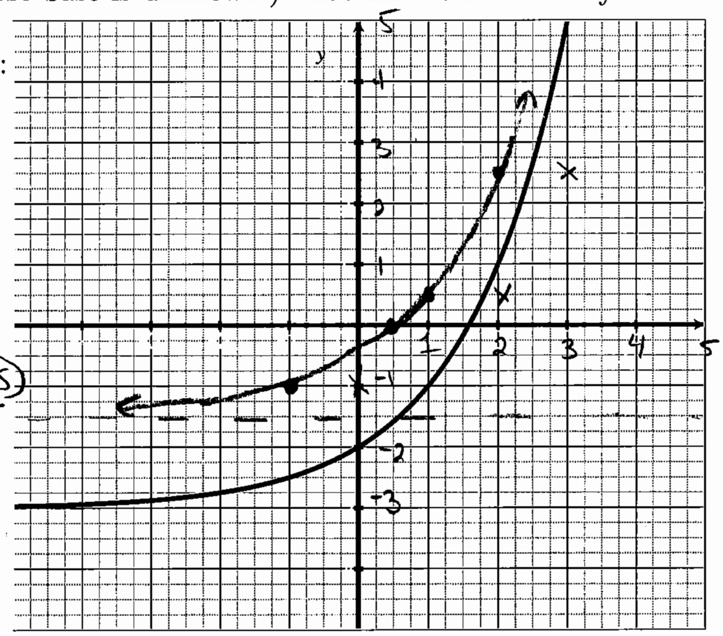
(b) [2] (Quiz3 #2) Find $\log(xz^2)$.

$\log(xz^2) = \log(x) + \log(z^2)$ prop (+1)

$= -2.5 + 6 = 3.5$ (+.5)

(+.5)

6. Let f be an exponential function (whose base is unknown!) that has been vertically shifted and graphed below.



(a) Estimate the following *if* possible:

i. [1] (Quiz3 #1) $f(2)$

1

ii. [2] (WrittenHW§3.2 #56) Range of f

y values (+.5)

$(-3, \infty)$ format (+.5)

(+.5) (+.5)

iii. [1] (LogFunctionActivity#1) Find a point on the graph of f^{-1}.

note $(2,1)$ is on f

$\Rightarrow (1,2)$ is on f^{-1}

iv. [1] all possible x such that $f(x) = -1$.

$x = 1$ note $f(1) = -1$

(b) [3] (LogFunctionsActivity #3) Find the formula/equation for f.

partial - treat 1.5 as a unit?

exp function vert shift

$\to b^x + v$

(+.5) (+.5)

Start (+.5)

normally thru $(0,1)$, now thru $(0,-2)$

$\Rightarrow$ vert. shift down 3 $\Rightarrow v = -3$ (+.5)

So $b^x - 3 = y$

Thru $(2,1) \Rightarrow b^2 - 3 = 1$ (+.5)

$b^2 = 4$ def (+.5)

$b = \pm 2$

So $y = 2^x - 3$ (+~~.5~~)

(c) [3] (WebHW3.1 #5) Graph $\frac{1}{2}f(x+1)$.

shape (+1)

vert shrink by 1/2 (+.5)

did (+.5)

horiz shift LEFT one unit (+.5)

2

did (+.5)

order does not matter

15

7. Solve for x:

(a) [3] (§3.4 #34) prop/logs (+1.5) $2 \cdot 3^{4x-5} - 6 = 8$

alg (+1)

$$2 \cdot 3^{4x-5} - 6 = 8 \quad +6 \; +6$$
$$\frac{2 \cdot 3^{4x-5}}{2} = \frac{14}{2}$$
$$3^{4x-5} = 7$$
$$\to 4x-5 = \log_3(7)$$
$$4x = \log_3(7) + 5$$
$$x = \frac{\log_3(7)+5}{4}$$ alg (+.5)

OR

$$2 \cdot 3^{4x-5} - 6 = 8 \quad +6 \; +6$$
$$\frac{2 \cdot 3^{4x-5}}{2} = \frac{14}{2}$$
$$3^{4x-5} = 7$$
$$\to \ln 3^{4x-5} = \ln 7$$
$$(4x-5)\ln 3 = \ln 7$$
$$4x-5 = \ln 7 / \ln 3$$
$$x = \left(5 + \frac{\ln 7}{\ln 3}\right) \div 4$$
$$\approx 1.693$$

(b) [3] (PracticeExam#8) $\log(x-1) = 2 - \log(x-16)$

log prop (+.5)

$$\log(x-1) + \log(x-16) = 2$$
$$\log (x-1)(x-16) = 2$$

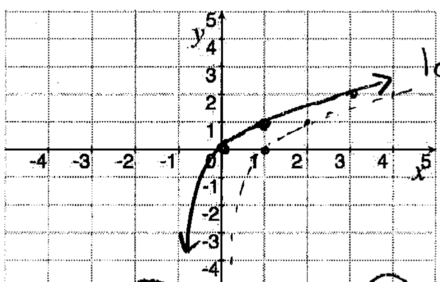
convert to exp (+1)

$$10^2 = (x-1)(x-16)$$
$$100 = x^2 - 17x + 16$$
$$\to x^2 - 17x - 84 = 0$$
$$(x-21)(x+4) = 0$$ solve quadratic (+1)

$\Rightarrow x = 21$ or $\cancel{-4}$ checked answers (+.5)

b/c domain

8. [4] (§3.1 #56) Provide a graph AND an algebraic rule/expression for: A logarithm function passing through $(0, 0)$.



shape (+1) thru (0,0) (+.5) match (+1)

log functions usually pass thru (1,0) so shift LEFT 1 unit

$$y = \log_2 (x+1)$$

logarithmic (+.5) shift (+1)

9. Consider a family who just welcomed their daughter. Assume the interest in this problem is 7.5% (the average S&P annual return over the last 20 years) compounded monthly.

(a) [3] (WebHWApplications1 #2) How much should guardians invest at the time their daughter is born to provide her with \$100,000 at the age of 18? (+.5) use $A = P\left(1+\frac{r}{n}\right)^{nt}$

(+.5) Find P so that

$$100{,}000 = P\left(1 + \frac{.075}{12}\right)^{12 \cdot 18} \to \frac{100{,}000}{3.84} = \frac{P \cdot 3.84}{3.84}$$

plug in values (+1)

$$\$26{,}033 = P$$ solve (+1)

(b) [4] (LogInPracticeActivity #1) If her guardians have \$50,000 now, how long until they can give their daughter \$100,000?

note: the "law of 72" could be used here?

(+.5) Find t so that

$$\frac{100{,}000}{50{,}000} = \frac{50{,}000\left(1 + \frac{.075}{12}\right)^{12t}}{50{,}000}$$

plug in values (+1)

alg (+.5)

$$2 = 1.00625^{12t}$$

use logs (+1)

$$\to \ln(2) = \ln(1.00625^{12t})$$
$$\ln 2 = 12t \ln(1.00625)$$
$$t = \frac{\ln 2}{12 \ln(1.00625)} \approx 9.3 \text{ years}$$ alg +1

17

27
23
50

10. Choose *ONE* of the following. Clearly identify which of the two you are answering and what work you want to be considered for credit.
No, doing both questions will not earn you extra credit.

   (a) (Workshop) A number of people in community who became infected during an epidemic t weeks after its outbreak is modeled by the logistic function $f(t) = \dfrac{30,000}{1+ae^{-kt}}$, where 30,000 people in the community are susceptible to the disease.

   i. [3] For a particular disease, contact tracers identify 5000 people initially infected and by the end of the fourth week there were 8280 people infected. Find the logistic function (the parameters a and k) that model this particular epidemic.

   ii. [1] Find the number of people who have been infected after six weeks.

   iii. [2] When will 75% of the population have become infected?

   (b) (Worksheet#3) Entropy, denoted as S, is a function of the number of possible states W, that are accessible to a system with a given amount of energy. We can explicitly compute entropy by

$$S = k\ln(W)$$

where k is Boltzmann's constant which is approximately $1.38065 \cdot 10^{-23} \mathrm{m^2 kg\ s^{-2} K^{-1}}$.

   i. [2] If a gas has $2 \cdot 10^{9000}$ possible states, what is the entropy of the gas?

   ii. [4] If liquid A has 10,000,000,000,000 (so $1 \cdot 10^{13}$) times more possible states than liquid B, which liquid has a higher entropy and what is the difference of the two entropys?

(a) i) find a and k given (0,5000) + (4,8280) (+.5)

$5000 = \dfrac{30,000}{1+a}$ (+.5) $\Rightarrow 5000(1+a) = 30,000$

$\Rightarrow 1+a = 6 \Rightarrow a = 5$ (+.5)

$8280 = \dfrac{30,000}{1+5e^{-k\cdot 4}}$ (+.5) $\Rightarrow 8280(1+5e^{-k4}) = 30,000$

(+1) $\Rightarrow 1+5e^{-4k} = 3.623 \Rightarrow 5e^{-4k} = 2.623$

$\Rightarrow e^{-4k} = .5246$

$\Rightarrow -4k = \ln(.5246) \Rightarrow k \doteq .161$

So $f(t) = \dfrac{30,000}{1+5e^{-.161t}}$

ii) find $f(6)$ plug in 6 (+.5)

$\dfrac{30,000}{1+5e^{-.161\cdot 6}} = 10,334$ people

got it (+.5)

iii) find t (+.5) when $.75 \cdot 30,000 = \dfrac{30,000}{1+5e^{-.161 \cdot t}}$ (+.5)

$(1+5e^{-.161t}).75 \cdot 30,000 = \dfrac{30,000}{.75 \cdot 30,000}$

(+1) alg

$1+5e^{-.161t} = 4/3$

$5e^{-.161t} = 1/3$

$e^{-.161t} = \frac{1}{15}$

$-.161t = \ln(1/15)$

$t \approx 16.8$ or 17 weeks

6

(b) i) find s given $W = 2 \cdot 10^{9000}$ (+.5) start (+.5)

$S = 1.38065 \cdot 10^{-23} \cdot \ln(2 \cdot 10^{9000})$ plug in (+1)

$= 1.38065 \cdot 10^{-23}(\ln 2 + \ln 10^{9000})$

$= 1.38065 \cdot 10^{-23}(\ln 2 + 9000 \ln 10)$

$\approx 2.8612 \cdot 10^{-19}$

ii) Let W_A be states of liquid A def (+1)

W_B be the states of liquid B

Know $W_A = 10^{13} W_B$ (+1)

$S_A - S_B = K\ln W_A - K \ln W_B$ (+.5)

(+.5) $= K(\ln(10^{13}W_B) - \ln W_B)$

alg (+1) $= K \ln\left(\dfrac{10^{13}\cancel{W_B}}{\cancel{W_B}}\right) = K \cdot \ln 10^{13}$

$= K \cdot 13 \cdot \ln 10$

So liquid A has a higher entropy + by (+.5)

$K \cdot 13\ln 10 \approx 4.13 \cdot 10^{-22}$