Practice TQS 120 Final

NAME: This is a sample final to be used for practice. This is *not a template* for the Final that will be given in class. Many of the questions on the Final will look quite di?erent than those appearing here.

[10] Let f & g, be functions.

Let $f(x) = x+1$ and $g(x) = 2x$

T F $(f \circ g)(x) = (g \circ f)(x)$ $(f \circ g)(x) = 2x+1$ but $(g \circ f)(x) = 2(x+1)$

T F $(\frac{f}{g})(x) = (\frac{g}{f})(x)$

T F $\sqrt{(x^2)} = x$ for all real numbers x. Let $x = -2$

T F If 2 is a root of g, then $g(2) = 0$.

T F $\ln \frac{x}{y} = \ln x - \ln y$ for all positive numbers x and y.

T F $\log(\log(10)) = 0$. $\log(1) = 0$ ✓

T F $\sin^{-1}(\sin x) = x$ for all real numbers x. $\sin^{-1}(\sin 5\pi/6) = \pi/6$

T F $\sin(\frac{\pi}{3} + x) = \sin \frac{\pi}{3} + x$ $\sin \pi/3 + x \,!\!= \sin(\pi/3) + x$

T F If $\cancel{\sin \theta > 0}$ and ~~$\tan \theta < 0$,~~ then $\cos \theta < 0$

T F The range of $\sin^{-1}$ is $[0, \pi]$

range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$

Right answers will *not* get credit without supporting work. Note "undefined" and "no solution" are possible answers.

1. Find all x such that

$$2(5 + (8 - x)^2)^{-\frac{1}{2}} - 1 = 0$$

$2(5+(8-x)^2)^{-1/2} = 1$

$(5+(8-x)^2)^{-1/2} = 1/2$

$\frac{1}{\sqrt{5+(8-x)^2}} = \frac{1}{2}$

$\frac{1}{5+(8-x)^2} = \frac{1}{4}$

$\rightarrow 1 = 1/4(5+(8-x)^2)$

$4 = 5+(8-x)^2$

$4-5 = (8-x)^2$

$-1 = (8-x)^2$

$\pm\sqrt{-1} = 8-x$

no real sol

2. [2] Explain what a function is.

A function takes inputs (the set of which is called a domain) to outputs (the set of which is called the range) in such a way that every input is sent to exactly one output.

3. Given $m(x) = \frac{x}{x-5}$, and $n(x) = \sqrt{4x-8}$,

(a) The function m passes the horizontal line test. Find m^{-1}.

$x = \frac{y}{y-5}$

$x(y-5) = y$

$xy - 5x = y$

$xy - y = 5x$

$y(x-1) = 5x$

$y = \frac{5x}{x-1}$

(b) [4] If $p(x) = 3m(x+1)$, find the domain and rule of p.

$3m(x+1) = 3\left[\frac{(x+1)}{(x+1)-5}\right] = \frac{3(x+1)}{x+1-5} = \frac{3x+3}{x-4}$

Domain: all x so that $x-4 \neq 0$ ie $x \neq 4$ or $(-\infty,4)\cup(4,\infty)$

(c) [3] Find the domain and rule of $n \circ m$.

$(n \circ m)(x) = n(m(x)) = n\left(\frac{x}{x-5}\right) = \sqrt{4\left(\frac{x}{x-5}\right)-8}$

Domain: all x so that $x-5 \neq 0$, $x \neq 5$ and $4\left(\frac{x}{x-5}\right) - 8 \geq 0$

$\frac{x}{x-5} \geq 8/4$

$\frac{x}{x-5} \geq 2$ consider

not ok | ok

10

$x \geq 2x-10$

$\cancel{x0} \geq \cancel{x10}$ $\Rightarrow x \geq 10$

(d) [5] Find the domain and rule of $\frac{n}{m}$.

$\left(\frac{n}{m}\right)(x) = \frac{\sqrt{4x-8}}{\left(\frac{x}{x-5}\right)}$

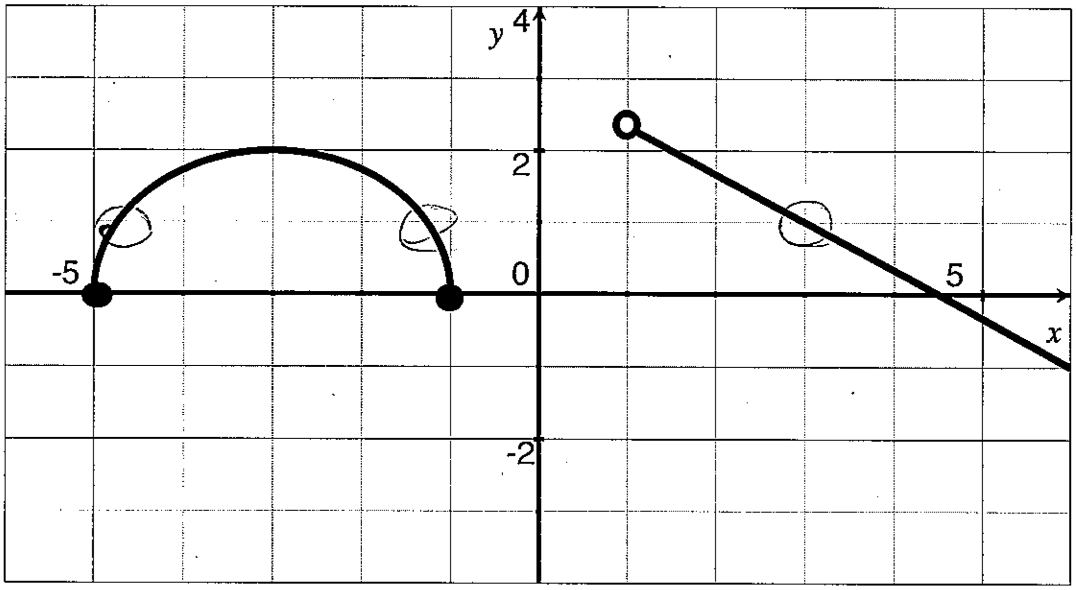
Domain:

$x-5 \neq 0$ And $\frac{x}{x-5} \neq 0$ and $4x-8 \geq 0$

$\Rightarrow x \neq 5$ $x \neq 0$ $4x \geq 8$, $x \geq 2$

or $[2,5)\cup(5,\infty)$

4. [3] Let the following be the graph of g.



(a) What is the domain of g?

$[-5,-1] \cup (1,6)$

or

$(1,\infty)$ (under b/c missing arrow)

(b) The function g is a piecewise defined function consisting of a straight line and a semicircle. Write down the rule for g.

semicircle w/ center at $(-3,0)$ and radius 2

$(x--3)^2+(y-0)^2=2^2$

$\Rightarrow (x+3)^2+y^2=4$

$\Rightarrow y=\sqrt{4-(x+3)^2}$

line through $(3,1)$ & $(6,-1)$

slope: $\frac{1--1}{3-6}=\frac{2}{-3}$

$1=-\frac{2}{3}(3)+b \rightsquigarrow b=3$

So

$$g(x)=\begin{cases}\sqrt{4-(x+3)^2} & -5\le x\le -1\\ -\frac{2}{3}x+3 & 1<x\end{cases}$$

(c) Use the graph above to *estimate* all x value(s) so that $g(x)=1$?

-4.6 $\quad -1.3$ $\quad$ and 3

(circled above)

this can be figured out exactly by solving for x in $1=\sqrt{4-(x+3)^2}$ and $1=-\frac{2}{3}x+3$

(d) Find the total length (of the curve and the line) that is graphed above.

total length = length of semicircle + length of line

$=\frac{1}{2}$(circumference of circle with radius 2) + $\sqrt{\text{line connecting } (6,1) \text{ to } (1,\frac{7}{3})}$

$=\frac{1}{2}(2\pi\cdot 2)+\sqrt{(6-1)^2+(1-\frac{7}{3})^2}$

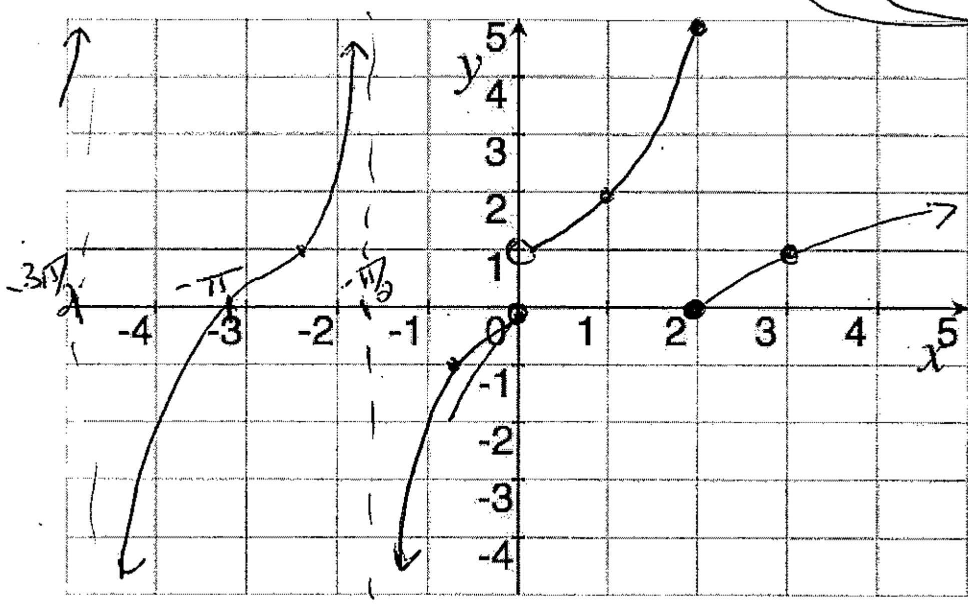
$=2\pi+\sqrt{25+3\frac{16}{9}}$

5. Define f by

$$f(x) = \begin{cases} \tan x & \text{if } x < 0 \\ x^2 + 1 & \text{if } 0 < x \le 2 \\ \log_2(x-1) & \text{if } x \ge 2 \end{cases}$$

shift right 1

note as written this is not a function b/c it fails the vert. line test.



(a) [8] Graph f on the axes above.

(b) [9] Find the following if possible:

$f(1)$

2

$f(2) + f(3)$

not defined b/c

$f(2) = 0$ AND 5.

$f(0)$

not defined

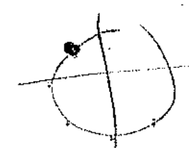
$f(\frac{-13\pi}{4}) = \tan\left(\frac{-13\pi}{4}\right)$

$= \tan\left(\frac{-8\pi}{4} - \frac{5\pi}{4}\right)$

$= \tan\left(-2\pi - \frac{5\pi}{4}\right)$

$= \tan\left(\frac{-5\pi}{4}\right)$

$= \dfrac{\sin\frac{-5\pi}{4}}{\cos\frac{-5\pi}{4}} = \dfrac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$



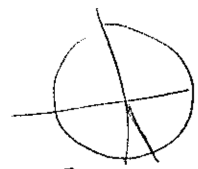
Range of f

$\mathbb{R}$

6. [8] Find all of the exact values x that satisfy the following:

$2\sin x = -\sqrt{3}$

$\sin x = \frac{-\sqrt{3}}{2}$



$x = \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\pi/3$

notice

$-\pi/3 + 2\pi,\ -\pi/3 + 2 \cdot 2\pi,\ -\pi/3 + k2\pi$

where k is an integer

are also solutions

Also $-2\pi/3$ & angles coterminal with it work.

$5^{4x-1} = 7^x$

$\ln 5^{4x-1} = \ln 7^x$

$(4x-1)\ln 5 = x \ln 7$

$x4\ln 5 - \ln 5 = x\ln 7$

$x4\ln 5 - x\ln 7 = \ln 5$

$x(4\ln 5 - \ln 7) = \ln 5$

$x = \frac{\ln 5}{4\ln 5 - \ln 7}$

7. [4] Find all exact values for x that satisfy the following:

$\log(x - 16) = 2 - \log(x - 1)$

$\log(x-16) + \log(x-1) = 2$

$\log(x-16)(x-1) = 2$

$(x-16)(x-1) = 10^2$

$x^2 - 17x + 16 = 100$

$x^2 - 17x - 84 = 0$

$(x-21)(x+4) = 0$

$x = 21$ or $\cancel{-4}$ b/c domain

$3^{5x}9^x = 27$

$3^{5x}(3^2)^x = 3^3$

$3^{5x}3^{2x} = 3^3$

$3^{5x+2x} = 3^3$ $\log_3$ both sides

$5x + 2x = 3$

$7x = 3$

$x = 3/7$

8. Simplify:

$\frac{\sqrt{c^2d^6}}{\sqrt{4c^3d^{-4}}} = \frac{(c^2d^6)^{1/2}}{(4c^3d^{-4})^{1/2}}$

$= \frac{(c^2)^{1/2}(d^6)^{1/2}}{4^{1/2}(c^3)^{1/2}(d^{-4})^{1/2}} = \frac{|c||d^3|}{2\, c^{3/2}\, d^{-2}}$

$= \frac{1}{2}|c|^{1-3/2}\, d^{3+2}$

$= \frac{1}{2}\, c^{-1/2}|d|^5$

(this would have been nicer if c & $d > 0$

$2 - \log_5(25z)$

$2 - \log_5 25z$

$\log_5 5^2 - \log_5 25z$

$\log \frac{25}{25z} = \log \frac{1}{z}$.

9. [7] Given $f(3) = 0$, use the factor theorem to find the other roots of $x^4 - 3x^3 - 25x^2 + 75x$

Since $f(3)=0$, 3 is a root

$\Rightarrow x-3$ is a factor

$$\begin{array}{r|l} & x^3 - 25x \\ x-3 & x^4 - 3x^3 - 25x^2 + 75x \\ & -(x^4 - 3x^3) \\ & \quad -25x^2 + 75x \\ & \quad -(-25x^2 + 75x) \\ & \qquad 0 \end{array}$$

So $\dfrac{x^4 - 3x^3 - 25x^2 + 75x}{x-3} = x^3 - 25x$

or

$$x^4 - 3x^3 - 25x^2 + 75x = (x-3)(x^3 - 25x)$$
$$= (x-3)x(x^2-25)$$
$$= (x-3)x(x+5)(x-5)$$

So the other roots are:

$0, -5 \,\&\, 5$

10. Simplify:

$\sin^{-1}(\sin \frac{3\pi}{4}) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = ?$

so that $\sin ? = 1/\sqrt{2}$ AND $-\pi/2 \le ? \le \pi/2$

$\pi/4$ works

$\Rightarrow ? = \frac{\pi}{4}$

$$\frac{\cos x}{1 - \sin x} + \frac{1 - \sin x}{\cos x}$$

$$\frac{(\cos x)^2 + (1-\sin x)^2}{(\cos x)(1-\sin x)}$$

Pyth $\dfrac{\cos^2 x + 1 - 2\sin x + \sin^2 x}{(\cos x)(1-\sin x)}$

$$\frac{1+1-2\sin x}{\cos x(1-\sin x)} = \frac{2-2\sin x}{\cos x(1-\sin x)}$$

$$= \frac{2}{\cos x}$$

11. [4] Let $\frac{-\pi}{2} < \theta < 0$ and $\cos\theta = \frac{1}{5}$. Find $\tan\theta$.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

we need $\sin\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + (1/5)^2 = 1$

$\sin^2\theta = 24/25$

$\sin\theta = \frac{\pm\sqrt{24}}{5}$

$\Rightarrow \sin\theta = \frac{-\sqrt{24}}{5}$ $\quad \tan\theta = \dfrac{\frac{-\sqrt{24}}{5}}{1/5} = -\sqrt{24}$

12. [6] Let $\frac{\pi}{2} < \phi < \pi$ and $\frac{-\pi}{2} < \theta < 0$. Given that $\sin\phi = \frac{2}{3}$ and that $\cos\theta = \frac{1}{5}$, find $\cos(\theta + \phi)$. (You are free to use results from #10 above.)

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$= \frac{1}{5}\cos\phi - \left(\frac{-\sqrt{24}}{5}\right)\left(\frac{2}{3}\right)$

from #11

$= \frac{1}{5}\cos\phi + \frac{2\sqrt{24}}{15}$

$= \frac{1}{5} \cdot \frac{-\sqrt{5}}{3} + \frac{2\sqrt{24}}{15}$

$= \frac{-\sqrt{5}}{15} + \frac{2\sqrt{24}}{15} \overset{6}{=} \frac{2\sqrt{24} - \sqrt{5}}{15}$

need to find $\cos\phi$

$\cos^2\phi + \sin^2\phi = 1$

$\cos^2\phi + (2/3)^2 = 1$

$\cos^2\phi = 1 - 4/9 = 5/9$

$\cos\phi = \frac{\pm\sqrt{5}}{3}$

$= \frac{-\sqrt{5}}{3}$

13. [5] Your given a 16 oz mocha that is a rather weak 3% espresso. You, knowing you'll be up late studying mathematics, would rather like a 30% espresso drink. Realizing this you purchase an espresso machine. How much weak mocha do you discard and replace with straight espresso to have a 16 oz mocha with the desired concentration?

Let x be the amount of mocha you keep (in oz)
let y be the amount of expresso you add. (in oz)

Note $x+y=16$.

espresso wanted in oz = espresso kept in oz + espresso added

$.3 \cdot 16 = .03 \cdot x + y$

We have 2 equations:

$x+y=16$

$.03x+y=4.8$

Solve for one variable in first: $x=16-y$

and the substitute that into the 2nd eq.

$.03(16-y)+y=4.8$

$\cancel{.48}-.03y+y=4.8$
$\cancel{-.48} \qquad -.48$

$.97y=4.32$

$y=\frac{4.32}{.97}$ oz

14. [5] Suppose a radioactive isotope is such that one-fifth of the atoms in a sample decay after three years. Find the half-life of this isotope

half life: $P_0\left(\frac{1}{2}\right)^{t/h}$

we know $\frac{1}{5}\cdot P_0$ decay so there are $\frac{4}{5}P_0$ still left in 3 years so we know

$\frac{\frac{4}{5}\cancel{P_0}}{\cancel{P_0}} = \frac{\cancel{P_0}\left(\frac{1}{2}\right)^{3/h}}{\cancel{P_0}}$ find h

$\frac{4}{5} = \left(\frac{1}{2}\right)^{3/h}$

$\ln \frac{4}{5} = \ln\left(\frac{1}{2}\right)^{3/h}$

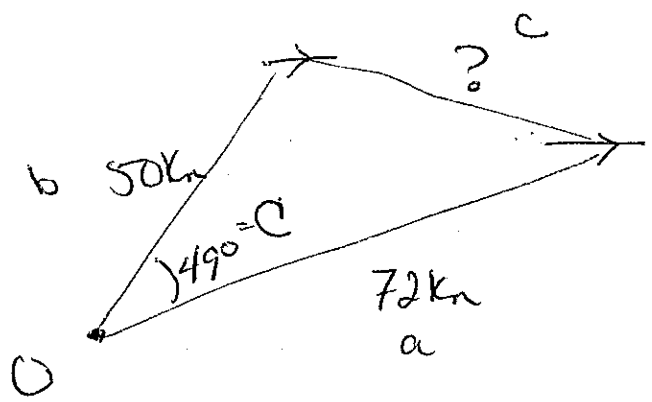
$\ln \frac{4}{5} = \frac{3}{h}\ln\frac{1}{2}$

$\frac{\ln \frac{4}{5}}{\ln \frac{1}{2}} = \frac{3}{h}$

$h\left(\frac{\ln \frac{4}{5}}{\ln \frac{1}{2}}\right) = 3$

$h = \frac{3\ln\frac{1}{2}}{\ln\frac{4}{5}}$

15. [5] An aircraft tracking station determines the distance from a common point O to each aircraft and the angle between the aicrafts. If angle O between the two aircrafts is equal to 49° and the distances from point O to the two aircrafts are 50km and 72km, find distance d between the two aircrafts.



SAS

L of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or

L of Cosides.

$$a^2 + b^2 - 2ab\cos C = c^2$$

Law of Cosines works better

$$72^2 + 50^2 - 2 \cdot 50 \cdot 72 \cos 49^\circ = c^2$$

$$\Rightarrow \quad c = \sqrt{72^2 + 50^2 - 100 \cdot 72 \cos 49^\circ}$$

$$= 54 \text{ km}$$ close?