$\#1 \to \#2, \#3$ $\quad$ $\#2 \to \#1$ $\quad$ $\#3 \to \#2, \#1$ $\quad$ $\#4 \to \#2, \#1$

8 min 20 pts $\quad$ 8 min 20 pts

#3: 7 min (10 pts)

95% of 20	19	of 10	9.5
85% of 20	17	of 10	8.5
75% of 20	15	of 10	7.5
60% of 20	12	of 10	6
0 of 20	0	of 10	0

(out of 50 points)

1) Determine if objects satisfy conditions

a) Does not have a commutative ring structure
(given ones have) $\quad$ Matrices, Group rings w/ non abelian group, $\mathbb{H}$

b) Has a subgroup that is not an ideal

✓ $\mathbb{Z}_{14}\times\mathbb{Z}_7$: $\langle(2,1)\rangle = \{(2,1),(4,2),(6,3),(8,4),(10,5),(12,6),(0,0)\}$
note $(4,2)\cdot(2,1) = (8,2) \notin \langle(2,1)\rangle$ not mult closed

✓ $\mathbb{Q}[[x]]$: $\langle 2\rangle = \{\ldots -2,0,2,4,\ldots\}$
note $x\cdot 2 \notin \langle 2\rangle$ no absorption

✓ $\mathbb{Z}_3\times\mathbb{Z}_3$: $\langle(1,1)\rangle = \{(1,1),(2,2),(0,0)\}$
note $(1,2)\cdot(1,1) = (1,2) \notin \langle(1,1)\rangle$ no absorption

X $\mathbb{Z}_{29}$ no subgroups

c) Is an integral domain:

X $\mathbb{Z}_{14}\times\mathbb{Z}_7$ eg $(1,0)\cdot(0,1)$

X $\mathbb{Z}_3\times\mathbb{Z}_3$ eg $(1,0)\cdot(0,1)$

✓ $\mathbb{Q}[[x]]$

✓ $\mathbb{Z}_{29}$ field

d) Infinite ring with at least 3 zero divisors (given ones do not satisfy) $\mathbb{Z}\times\mathbb{Z}$

e) Is not a field ✓ $\mathbb{Z}_{14}\times\mathbb{Z}_7$, ✓ $\mathbb{Q}[[x]]$, ✓ $\mathbb{Z}_3\times\mathbb{Z}_3$ $\quad$ X $\mathbb{Z}_{29}$

2) Prove or provide a counterex. for "The ideal $I = (x, 5)$ is maximal in $\mathbb{Z}[x]$."

Proof: Let M be an ideal in $\mathbb{Z}[x]$ $\ni$ $I \subsetneq M \leq \mathbb{Z}[x]$.

If $M = \mathbb{Z}[x]$ then I will be maximal.

It suffices to show $\exists$ a unit in M to imply $M = \mathbb{Z}[x]$.

Recall b/c $1 \in \mathbb{Z}[x]$ the elements of I are

$I = \{5a + xb \mid a, b \in \mathbb{Z}[x]\}$. Thus all const. are divisible by 5.

If $I \subsetneq M$ then $\exists$ $p(x) \in M$ with constant term 1, 2, 3 or 4. when divided by 5. Let $p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$

Case 1: $\alpha_0 = 1 \bmod 5$ so $\exists k \in \mathbb{Z} \ni \alpha_0 = 1 + k \cdot 5$

Note $\alpha_1 x + \dots + \alpha_n x^n \in I \subseteq M$ and $k5 \in I \subseteq M$

Consider $p(x) - (k5) - (\alpha_1 x + \dots + \alpha_n x^n) = 1$

and is in M b/c M is a subgroup. $\therefore M = \mathbb{Z}[x]$.

Case 2: $\alpha_0 = 2 \bmod 5$ so $\exists k \in \mathbb{Z} \ni \alpha_0 = 2 + k5$

Observe $2p(x) = 2(2 + k5 + \alpha_1 x + \dots + \alpha_n x^n)$

Note $2(\alpha_1 x + \dots + \alpha_n x^n) \in I \subseteq M$ and $5 + 10k \in I \subseteq M$

Consider $2p(x) - 2(\alpha_1 x + \dots + \alpha_n x^n) - (5 + 10k) = -1$

and is in M b/c M is a subgp $\therefore M = \mathbb{Z}[x]$

All the cases are similar - really just need to use $\gcd(i, 5) = 1$

Case 3 $\alpha_0 = 3 \bmod 5$ so $\exists k \in \mathbb{Z} \ni \alpha_0 = 3 + k5$

Use $2p(x) - (k+1)5 - 2(\alpha_1 x^1 + \dots + \alpha_n x^n) = 1 \in \mathbb{Z}[x]$

Case 4: $\alpha_0 = 4 \bmod 5$ so $\exists k \in \mathbb{Z} \ni \alpha_0 = 4 + k5$

Use $p(x) - (k+1)5 - (\alpha_1 x^1 + \dots + \alpha_n x^n) = 1 \in \mathbb{Z}[x]$

Note $\mathbb{Z}[x]/(x, 5)$ is a field

#3) $\mathbb{F}_3[x]$ $\quad 3 \cdot 3 \cdot 2 = 18$ options

$0 + 0x + 1x^2 = (x-0)(x-0)$

$0 + 0x + 2x^2 = 2(x-0)(x-0)$

$\boxed{1 + 0x + 1x^2}$

$1 + 0x + 2x^2 = (x-1)(2x+2)$

$2 + 0x + 1x^2 = (x-1)(x+1)$

$\boxed{2 + 0x + 2x^2}$

$1 + 1x + 1x^2 = (x-1)(x+2)$

$1 + 2x + 1x^2 = (x-2)(x+1)$

$\boxed{1 + 1x + 2x^2}$

$\boxed{1 + 2x + 2x^2}$

$0 + 1x + 1x^2 = (x-0)(1+x)$

$0 + 2x + 1x^2 = (x-1)(2+x)$

$0 + 1x + 2x^2 = (x-0)(1+2x)$

$0 + 2x + 2x^2 = 2(x-0)(1+x)$

$\boxed{2 + 1x + 1x^2}$

$\boxed{2 + 2x + 1x^2}$

$2 + 1x + 2x^2 = (x-2)(2x+2)$

$2 + 2x + 2x^2 = 2(x-1)(x+2)$

Perform • addition $\quad (2 + x + (f(x))) + (2 - 2x + (f(x)))$

• multiplication : $\quad (x + (f(x))) \cdot (x + f(x))$

• (other mult)

$\mathbb{F}_3[x]$ field? $\quad$ how do you know?

ID?

4) $R = \mathbb{Z}[\sqrt{3}]$ and $\mathrm{Frac}(R)$ be the field of fractions

$$\mathrm{Frac}(R) = \left\{ \frac{\alpha}{\beta} \;\middle|\; \alpha \in \mathbb{Z}(\sqrt{3}), \beta \in \mathbb{Z}(\sqrt{3}) \setminus \{0\} \right\}$$

$$= \left\{ \frac{a+b\sqrt{3}}{c+d\sqrt{3}} \;\middle|\; a,b,c,d \in \mathbb{Z},\ c \wedge d \neq 0 \right\}$$

$$= \left\{ \frac{(a+b\sqrt{3})(c-d\sqrt{3})}{c^2-d^2 3} \;\middle|\; a,b,c,d \in \mathbb{Z},\ c \wedge d \neq 0 \right\}$$

$$= \left\{ \frac{(ac-bd3)+(-ad+bc)\sqrt{3}}{c^2-3d^2} \;\middle|\; a,b,c,d \in \mathbb{Z},\ c \wedge d \neq 0 \right\}$$

is $\frac{1}{2} \in \mathrm{Frac}(R)$? is $(\sqrt{3})^{-1} \in \mathrm{Frac}(R)$? is $\sqrt{6} \in \mathrm{Frac}(R)$?

$$\frac{1}{2} + (\sqrt{3})^{-1} = \frac{1}{2} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}+2}{2\sqrt{3}} = \frac{2+\sqrt{3}}{0+2\sqrt{3}}$$

mult, divide, subtract

find another representation for $5+2\sqrt{3}$

Last Problem:

1) well defined check
group homomorphism
mult homomorphism note $\varphi(1 \cdot 1) = \varphi(1)\varphi(1) = \varphi(1)$
image needs to be idempotent.

2) one-to-one ring homom that is not onto?

3) onto ring homom that is not one-to-one

4) what $\mathbb{Z}_{10 \cdot a} / \ker(\phi) \cong b$ - (show how to build iso)

idempotent elements: $\mathbb{Z}_{10}$: 1, 5, 6

$\mathbb{Z}_{20}$: 1, 5, 16

$\mathbb{Z}_{30}$: 1, 6, 10, 15, 16, 21, 25

$\mathbb{Z}_{40}$: 1, 16, 25

$\mathbb{Z}_{50}$: 1, 25, 26

$\mathbb{Z}_{60}$: 1, 16, 21, 25, 36, 40, 45