Key

Midterm TMath 402 Winter 2026

Recall you may use a one-sided 8.5" by 11" sheet of notes (or surface with equal area) with what ever you like written or typed on it. Additionally, a non-internet accessing calculator, or Desmos Test Mode can be used.

Show your work for the following problems to earn full marks.

1. [3] Prove or provide a counterexample for the following statement: If G is cyclic then all generating sets of G consist of one and only one element.

ogic (+1)
generating set (+1)
notation (+.5)

False (+.5) Example is $C_{10} = \langle 1 \rangle$ and $C_{10} = \langle 2, 5 \rangle$

The generating set $\{2, 5\}$ has 2 elements

note: I should have said ... then all ~~minimal~~ generating sets of G...

2. [3] Prove or provide a counterexample for the following statement: Let S_{10} be the permutation group on ten elements. For all $\sigma \in S_{10}$, $\sigma^{10} = ()$, where $()$ is the identity permutation.

gic (+.5)
order (+1)
permutations (+.5)
notation (+.5)

False (+.5) Consider the element $\sigma = (123)(4567)$ or $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 & 8 & 9 & 10 \end{bmatrix}$

The order of σ is gcd$(3,4)$ or 12

$\Rightarrow \sigma^{10} \neq (1)$ in particular

$\sigma^{10} = (123)(46)(57)$

3. [3] Prove or provide a counterexample for the following statement: If G is a group with elements σ and τ, then $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

(+.5) [This is the socks and shoes theorem?

gic (+1)
inverse def (+1)
notation (+.5)

We verify that $\tau^{-1}\sigma^{-1}$ is the inverse of $\sigma\tau$ by examining $(\sigma\tau)(\tau^{-1}\sigma^{-1})$ and $(\tau^{-1}\sigma^{-1})(\sigma\tau)$

$= \sigma(\tau\tau^{-1})\sigma^{-1}$ $\qquad = \tau^{-1}(\sigma^{-1}\sigma)\tau$ b/c assoc

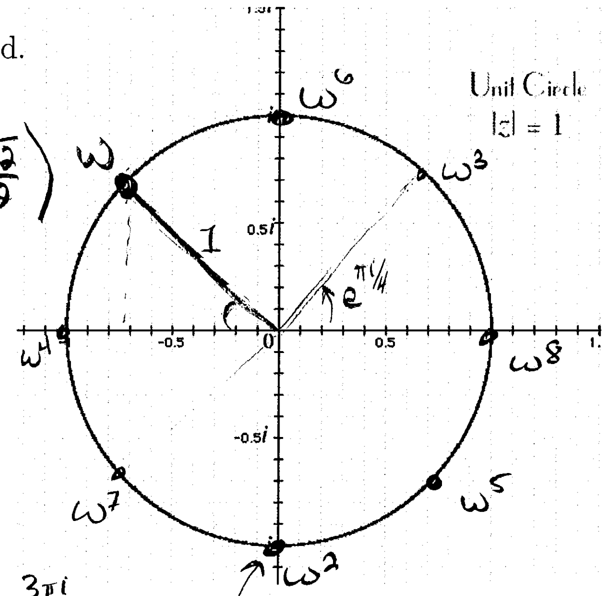
$= \sigma e \sigma^{-1}$ $\qquad = \tau^{-1} e \tau$ b/c $\tau\tau^{-1} = e = \sigma\sigma^{-1}$

$= \sigma\sigma^{-1}$ $\qquad = \tau^{-1}\tau$ b/c e is identity

$= e$ $\qquad = e$

4. Consider the group $G = \langle \omega \rangle$ in $\mathbb{C}^*$ where $\omega = e^{\frac{3\pi i}{4}}$

(a) [1] Plot ω on the complex plane provided.



Signs (+.5)

(b) [2] What are the rectangular coordinates of ω? $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ ω

(+.5) [Sohcahtoa

(+.5) [$\cos(\pi/4) = \frac{x}{h}$ where $x < 0$

$\Rightarrow x = \cos(\pi/4) = \frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$

(+.5) [Similarly $y = \sin(\pi/4) = \frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$

(c) [2] Find ω^2. This could be plotted or written in coordinates (polar, rectangular, etc)

$$\omega^2 = \left(e^{\frac{3\pi i}{4}}\right)^2 = e^{2\left(\frac{3\pi i}{4}\right)} = e^{\frac{3\pi i}{2}}$$

or $\left(\cos\frac{3\pi}{2}, \sin\frac{3\pi i}{2}\right)$

or $(0, -1)$ or

(d) [2] What is the order of ω?

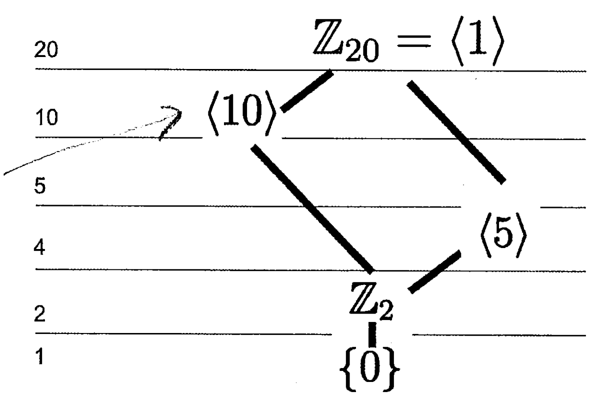
8 tracing around unit circle or find k so $e^{\left(\frac{3\pi i}{4}\right)k} =$

(e) [1] What is the order of G?

8 since ω generates C_8

notation (+.5)
start (+.5)

5. [4] The subgroup lattice for $\mathbb{Z}_{20}$ shown on the right is incorrect. Identify at least three errors.

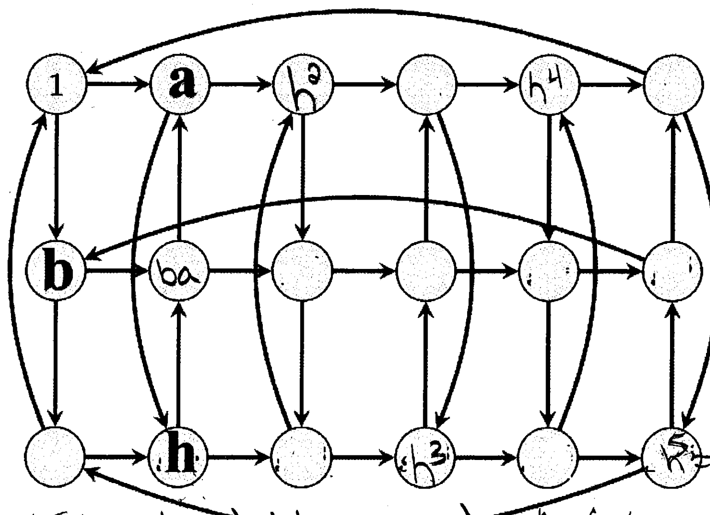


(+1) 1) The subgroup generated by 10 is $\{10, 0\}$ which has order 2, so $\langle 10 \rangle$ is in the wrong spot.

(+1) 2) There are some missing subgroups - you can tell b/c there are no subgps of order 5 but $5 | 20$ and a Thm from class tells us for cyclic groups if $k | n$ then $\exists$ a subgroup of order k. In this case it is $\langle 4 \rangle = \{4, 8, 12, 16, 0\}$

3) ...

6. Use the provided Cayley graph of a group H to answer the following questions:



(a) [1] How many generators are used in the Cayley graph?

2 a and b (red) (blue)

(b) [2] Is H abelian? Justify your answer.

abelian def (+.5)

nope (+.5)

(+1) [$ba \neq h = ab$

(c) [2] Does h generate H? Justify your answer.

notation (+.5)

(+.5) [no | the order of h is 6 (I traced ~~red/blue around~~ on the Cayley graph)

(+1) | Note $b \notin \langle h \rangle$ so $\langle h \rangle \neq H$.

(d) [2] Find h^{-1} in terms of the generators.

inv. def (+.5)

use a's + b's (+.5)

just one (+1)

~~$h = ab$~~

~~so $h^{-1} = (ab)^{-1} = b^{-1}a^{-1}$~~

question wanted to use a's + b's

or $h^6 = h(h^5) = e \Rightarrow h^5 = h^{-1}$ so $h^{-1} = ababababab$ | a^5b OR -OR- | b^2a^5

7. Consider the work provided by coPilot to prove if $\gcd(n, m) = 1$, then $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$.

(+.5) hard to know b/c outside scope

(a) [2] Are all the facts and data correct/does it align with the class?

(+.5) No, the class has not talked about homomorphisms or isomorphisms (+1)

(b) [2] Provide concrete suggestions for how to improve the reading level or tone.

An intro would be nice, like "we will build an isomorphism", 2 some kind of guidance for the reader really needs to be added.

Proof. Define a homomorphism, $\phi : \mathbb{Z}_{nm} \to \mathbb{Z}_n \times \mathbb{Z}_m$ by $\phi([k]_{nm}) = ([k]_n, [k]_m)$.

- Homomorphism: $\phi([k + l]_{nm}) = ([k + l]_n, [k + l]_m) = ([k]_n + [l]_n, [k]_m + [l]_m)$. ← words please
- Kernel: $\phi([k]_{nm}) = (0, 0)$ if and only if $k \equiv 0 \mod n$ and $k \equiv 0 \mod m$. So k is a multiple of both n and m, hence a multiple of $\text{lcm}(n, m)$. Thus

$$\ker \phi = \{[k]_{nm} | \text{lcm}(n, m) \text{ divides } k\},$$

which has size

$$|\ker \phi| = \frac{nm}{\text{lcm}(n, m)}.$$

Since $\gcd(n, m) = 1$, the $\text{lcm}(n, m) = nm$ so $|\ker \phi| = \frac{nm}{nm} = 1$.

Since $|\mathbb{Z}_{nm}| = nm = |\mathbb{Z}_n \times \mathbb{Z}_m|$, an injective homomorphism between finite groups of the same order is an isomorphism.

□

8. [8] Prove the following: Let G be a group and fix an $x \in G$. The centralizer of x, denoted $C(x) = \{g \in G | gx = xg\}$, is a subgroup of G.

se the
Rubric (+4)
rom
written HW

d subg (+1.0)
otation (+1)
ganization (+1.5)
f each piece (+1.5)

We will verify that $C(x)$ satisfies the group properties:

Associativity: Note $C(x) \subseteq G$ and G is a group so we get to inherit associativity from G.

Identity: We want to verify the identity e, satisfies the condition to be in $C(x)$. Note b/c e is an identity in G,

$ex = x = xe$ so $e \in C(x)$.

Inverse: Let $h \in C(x)$, we WTS $h^{-1} \in C(x)$.

That is we WTS $h^{-1}x = xh^{-1}$.

Note $h \in C(x)$ so we know $hx = xh$.

Apply h^{-1} to both sides (on left and right): $h^{-1}hxh^{-1} = h^{-1}xhh^{-1}$

$\Downarrow$

Simplifying b/c $h^{-1}h = e = hh^{-1}$ $\quad exh^{-1} = h^{-1}xe$

Since e is the identity $\quad xh^{-1} = h^{-1}x$

Or by symmetry of equal sign $\quad h^{-1}x = xh^{-1}$

which is what we wanted to show.

Closure: Let $h_1, h_2 \in C(x)$, we WTS $h_1h_2 \in C(x)$.

That is we WTS $(h_1h_2)x = x(h_1h_2)$.

Note $(h_1h_2)x = h_1(h_2x)$ by assoc

$= h_1(xh_2)$ b/c $h_2 \in C(x)$ $[h_2x = xh_2]$

$= (h_1x)h_2$ by assoc

$= (xh_1)h_2$ b/c $h_1 \in C(x)$ $[h_1x = xh_1]$

$= x(h_1h_2)$ by assoc.