Take Home Midterm Section Key

$H \leq S_8$ with $H = \langle (1234)(5876), (1537)(2648) \rangle$. Let $a = (1234)(5876)$ and $b = (1537)(2648)$

1) [2] Is H abelian? Provide justification.

It suffices to check the generators:

$ab = (1638)(2745) \neq (1836)(2547) = ba$

Since $ab \neq ba$, H is not abelian.

abelian def (+.5)
computations (+1)
got it (+.5)

2) [3] Provide a cycle graph for the subgroup H so the orbits of the elements can easily be seen.

Need to get a feel for how the elements act:

$a^2 = (13)(24)(57)(68)$ $\quad a^3 = (1432)(5678) \quad a^4 = (1)(2)(3)(4)(5)(6)(7)(8) = () = e$

$\| \qquad\qquad\qquad\qquad \|$

$b^2 = (13)(24)(57)(68)$ $\quad b^3 = (1735)(2846) \quad b^4 = (1)(2)(3)(4)(5)(6)(7)(8) = () = e$

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$(ab)^2 = (13)(24)(57)(68)$ $\quad (ab)^3 = (1836)(2547) = ba \quad (ab)^4 = (1)(2)(3)(4)(5)(6)(7)(8) = () = e$

$a(ab) = (1735)(2846) = b^3$

$(ab)(a) = (1537)(2648) = b$

$b(ab) = (1234)(5876) = a$

$(ab)b = (1432)(5678) = a^3$

$a^2(ab) = (ab)^2(ab) = (ab)^3$
$= b^2(ab) = (ab)b^2$
$= (ab)a^2$

not getting any new elements $\Downarrow$ have only 8 elements

relations found: (+1)
$a^4 = b^4 = (ab)^4 = e$
$a^2 = b^2 = (ab)^2$

(+.5) distinct elements:
a, a^3
b, b^3
$ab, (ab)^3$
e, a^2

computations (+.5)

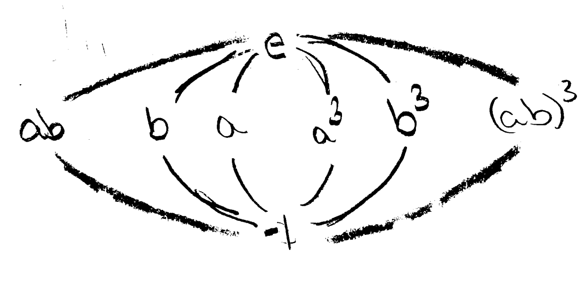
We know all the groups of order 8:

$\underbrace{\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}_{\text{abelian so not } H}, D_4, Q_8$

$D_4 \downarrow$ only 2 elements of order 4

$Q_8 \downarrow$ has 3 elements of order 4 $a, b \,\&\, ab$

Cycle Graph: (+1)



3) [3] Create the subgroup lattice for H

all subgroups (+1)
containments (+1)
formatting (+1)

$$\begin{tikzcd}[column sep=small]
\text{Order} & & & \\
8 & & H \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
4 & \langle a \rangle \arrow[dr, dash] & \langle ab \rangle \arrow[d, dash] & \langle b \rangle \arrow[dl, dash] \\
2 & & \langle a^2 \rangle \arrow[d, dash] & \\
1 & & \langle e \rangle &
\end{tikzcd}$$

4) [2] Identify what group in the textbook the subgroup H has the same structure of
Provide some justification:

Since $|H|=8$ we consider all groups of order 8. The text lists all of these: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 and Q_8.
The first 3 are abelian & #1 indicated H was not abelian.
Thus we consider D_4 and Q_8.
Recall the order of the elements in D_4 are:

element	e	r	r^2	r^3	f	rf	r^2f	r^3f
order	1	4	2	4	2	2	2	2

In H we have six elements with order 4 $(a, a^3, b, b^3, \text{etc})$
So H is not D_4.
Process of elimination leaves Q_8 (+.5) & notice the order of the elements do match up:

element	e	i	$-i$	j	$-j$	k	$-k$	-1
order	1	4	4	4	4	4	4	2
in H	e	a	a^3	b	b^3	ab	$(ab)^3$	a^2

justification (+1.5)