Midterm TMath 402 Winter 2026

Recall you may use a one-sided 8.5" by 11" sheet of notes (or surface with equal area) with what ever you like written or typed on it. Additionally, a non-internet accessing calculator, or Desmos Test Mode can be used.

Show your work for the following problems to earn full marks.

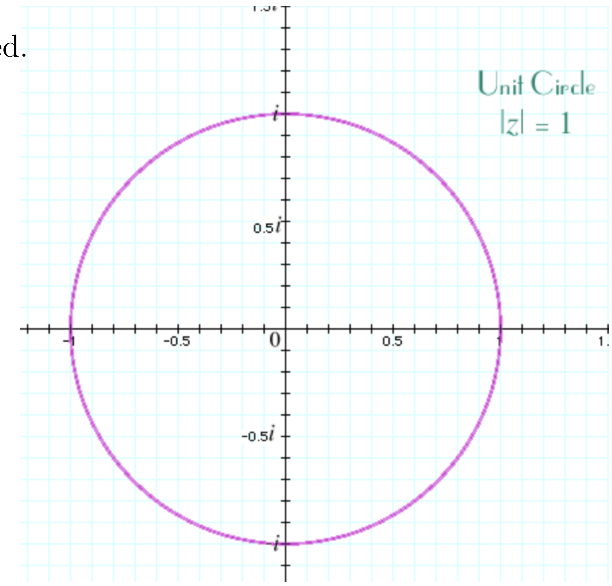
1. [3] Prove or provide a counterexample for the following statement: If G is cyclic then all generating sets of G consist of one element.

2. [3] Prove or provide a counterexample for the following statement: Let S_{10} be the permutation group on ten elements. For all $\sigma \in S_{10}$, $\sigma^{10} = ()$, where $()$ is the identity permutation.

3. [3] Prove or provide a counterexample for the following statement: If G is a group with elements σ and τ, then $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

4. Consider the group $G = \langle \omega \rangle$ in $\mathbb{C}^*$ where $\omega = e^{\frac{3\pi i}{4}}$

(a) [1] Plot ω on the complex plane provided.



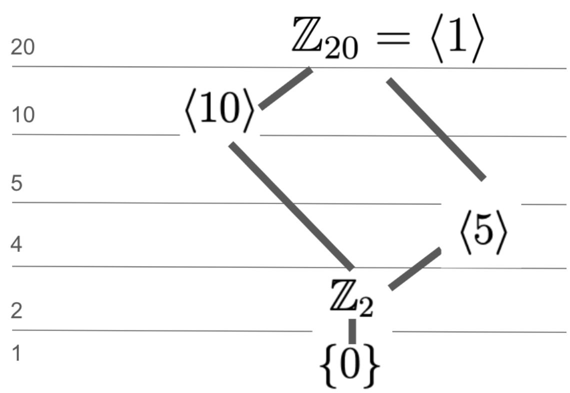
(b) [2] What are the rectangular coordinates of ω?

(c) [2] Find ω^2. This could be plotted or written in coordinates (polar, rectangular, etc)

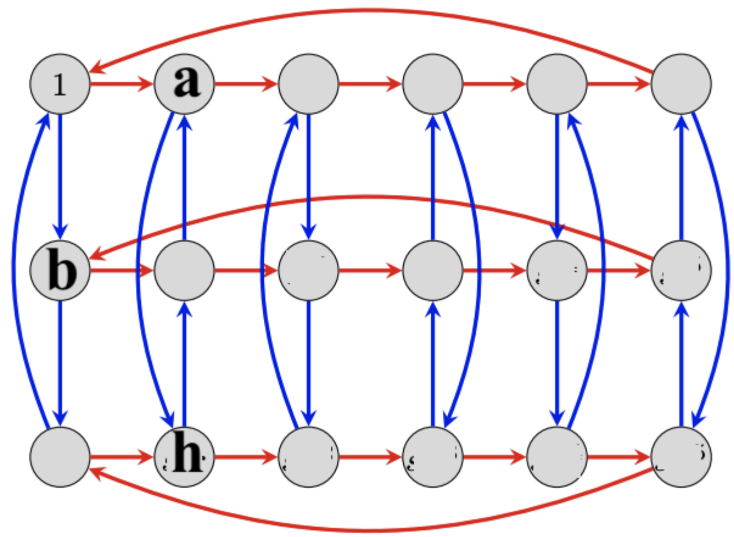
(d) [2] What is the order of ω?

(e) [1] What is the order of G?

5. [4] The subgroup lattice for $\mathbb{Z}_{20}$ shown on the right is incorrect. Identify at least three errors.



6. Use the provided Cayley graph of a group H to answer the following questions:



(a) [1] How many generators are used in the Cayley graph?

(b) [2] Is H abelian? Justify your answer.

(c) [2] Does h generate H? Justify your answer.

(d) [2] Find h^{-1} in terms of the generators.

7. Consider the work provided by coPilot to prove if $\gcd(n, m) = 1$, then $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$.

(a) [2] Are all the facts and data correct/does it align with the class?

(b) [2] Provide concrete suggestions for how to improve the reading level or tone.

Proof. Define a homomorphism, $\phi : \mathbb{Z}_{nm} \to \mathbb{Z}_n \times \mathbb{Z}_m$ by $\phi([k]_{nm}) = ([k]_n, [k]_m)$.

- Homomorphism: $\phi([k+l]_{nm}) = ([k+l]_n, [k+l]_m) = ([k]_n + [l]_n, [k]_m + [l]_m)$.
- Kernel: $\phi([k]_{nm}) = (0, 0)$ if and only if $k \equiv 0 \mod n$ and $k \equiv 0 \mod m$. So k is a multiple of both n and m, hence a multiple of $\text{lcm}(n, m)$. Thus

$$\ker \phi = \{[k]_{nm} | \text{lcm}(n, m) \text{ divides } k\},$$

which has size

$$|\ker \phi| = \frac{nm}{\text{lcm}(n, m)}.$$

Since $\gcd(n, m) = 1$, the $\text{lcm}(n, m) = nm$ so $|\ker \phi| = \frac{nm}{nm} = 1$.

Since $|\mathbb{Z}_{nm}| = nm = |\mathbb{Z}_n \times \mathbb{Z}_m|$, an injective homomorphism between finite groups of the same order is an isomorphism.

□

8. [8] Prove the following: Let G be a group and fix an $x \in G$. The centralizer of x, denoted $C(x) = \{g \in G | gx = xg\}$, is a subgroup of G.