$\pi_0(X)$ & Reeb Spaces

While working in a group make sure you:

- Expect to make mistakes but be sure to reflect/learn from them!
- Are civil and are aware of your impact on others.
- Assume and engage with the strongest argument while assuming best intent.

Definition 0.1. A *path* in a topological space, X, is a continuous function $\gamma : [0,1] \to X$. We say γ is a path from $\gamma(0)$ to $\gamma(1)$.

Definition 0.2. Let X be a topological space. The path components of X, denoted $\pi_0(X)$, is the space $X/\sim_C$ where $a \sim_C b$ if there is a path from a to b.

1. Let $\Gamma_1 = (V_1, E_1)$ where $V_1 = \{v_i\}_{i\in[6]}$ and $E_1 = \{v_1v_2, v_1v_3, v_2v_3, v_4v_5, v_4v_6, v_5v_6\}$. Assume Γ_1 is realized/embedded in $\mathbb{R}^3$.

   (a) Create a path from v_1 to v_3, if possible.

   (b) Find a vertex $\sim_C$ to v_1

   (c) Find $|\pi_0(\Gamma_1)|$

2. Let $\Gamma_2 = (V_2, E_2)$ where $V_2 = \{w_i\}_{i\in[6]}$ and $E_2 = \{w_1w_2, w_1w_3, w_2w_4, w_3w_5, w_4w_6, w_5w_6\}$. Assume Γ_2 is realized/embedded in $\mathbb{R}^3$.

   (a) Create a path from w_1 to w_4, if possible.

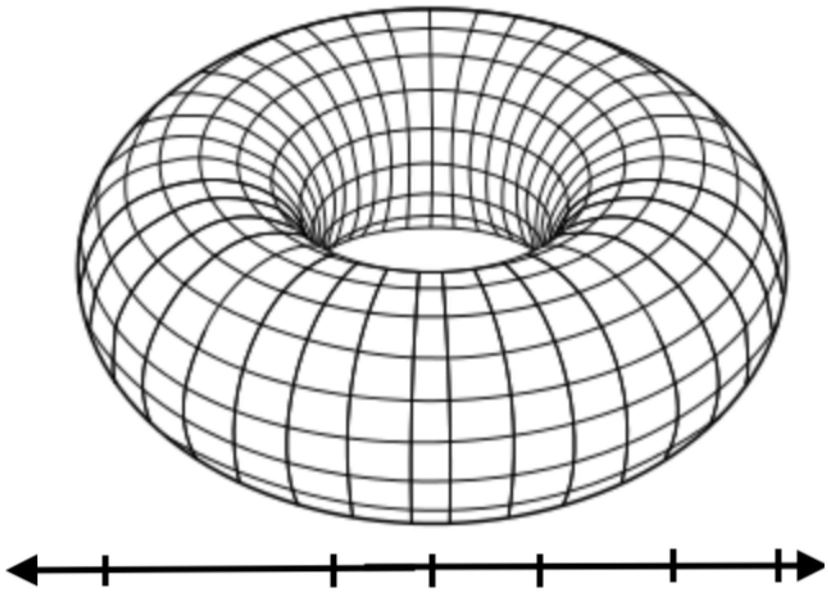
   (b) Find a vertex $\sim_C$ to w_2

   (c) Find $|\pi_0(\Gamma_2)|$

3. Consider the continuous map $f : \mathbb{R} \to \mathbb{R}$ (where $\mathbb{R}$ has the usual topology) defined by $f(x) = (x+3)(x+1)(x-2)(x-4)$. Find $|\pi_0(f^{-1}(\mathcal{B}_{.01}(0)))|$.

Definition 0.3. Let X be a topological space and $f : X \to \mathbb{R}$ a continuous function. We construct an equivalence relation on X as follows: for each $c \in \mathbb{R}$ and $x,y \in f^{-1}(c)$, $x \sim y$ if they are in the same path component of $f^{-1}(c)$. The *Reeb space* of f is $X/\sim$.

4. Let X be the torus in $\mathbb{R}^3$ as shown in the picture below. Let $f : X \to \mathbb{R}$ defined by $f(x, y, z) = x$. Sketch the Reeb space of f.



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