Proofs

the mostly algebraic kind

1. If m and n are both perfect squares, then nm is also a perfect square.

1. For each of the following "Theorems", determine:

   (a) if the "Theorem" is true, and
   (b) if the "Proof" is valid.

   Theorem 1. *If n is an odd integer, then n^2 is odd.*

   Proof. Assume that n is an odd integer. We want to show that n^2 is odd.

   Since n is an odd integer, there exists an integer a, so that $n = 2a+1$. Using algebra we see,

   $$n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1.$$

   By definition of an odd integer we see n^2 is an odd integer. □

   Theorem 2. *If n^2 is positive, then n is positive.*

   Proof. Assume that n^2 is positive. We want to show that n is positive.

   Consider if n is positive, then $n > 0$. We can multiply this inequality by the positive number n on both sides and arrive at $n^2 > 0n$ or $n^2 > 0$. Thus, if n is positive we know n^2 is positive.

   Since we were assuming that n^2 was positive, we can conclude from the above that n is positive. □

   Theorem 3. *Let m, n, and p be integers. If $m+n$ and $n+p$ are even integers, then $m+p$ is an even integer.*

   Proof. Assume that $m+n$ and $n+p$ are even integers. We want to show that $m+p$ is an even integer.

   Since $m+n$ is an even integer, there exists integers a and b such that

   $$m + n = 2a + 2b.$$

   Thus we know that $m = 2a$ and $n = 2b$.

   Since $n+p$ is an even integer and $n = 2b$, we know that there exists and integer c such that

   $$n + p = 2b + 2c.$$

   Thus we also know that $p = 2c$.

   Now we can consider $m+p$ which equals $2a + 2c = 2(a+c)$. Thus $m+p$ is even. □

   Theorem 4. *If n is an integer and n^2 is odd, then n is odd.*

   Proof. We will show the contrapositive of the statement, that is, if n is not odd, we will show that either n is not an integer or n^2 is not odd. Said more simply, we will show, if n is even, then either n wasn't an integer or n^2 is even.

   Since n is even, we know there exists an integer a such that $n = 2a$. Using algebra we see,

   $$n^2 = (2a)^2 = 4a^2 = 2(2a^2),$$

   thus, by definition of an even integer, n^2 is even. □