Induction

1. Determine if the following proofs are valid or not.
   If not valid, highlight the fallacies/error in logic.

Theorem 1. *For every positive integer n,* $\sum_{i=2}^{n} \frac{1}{(i-1)i} = \frac{3}{2} - \frac{1}{n}$

Proof 1. *The result is true when $n = 1$ because* $\frac{1}{1 \cdot 2} = \frac{3}{2} - \frac{1}{1}$.

Assume that the result is true for n. Then

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{(n-1) \cdot n} + \frac{1}{n(n+1)} = \frac{3}{2} - \frac{1}{n} + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{3}{2} - \frac{1}{n+1}$$

Hence, the result is true for $n+1$ if it is true for n. Thus we conclude $\sum_{i=2}^{n} \frac{1}{(i-1)i} = \frac{3}{2} - \frac{1}{n}$.

Theorem 2. *Every set of lines in the plane, no two of which are parallel, meet in a common point.*

Proof 2. *Certainly two lines that are not parallel meet in a common point.*

Assume a collection of k lines, no two of which are parallel, meet at a common point. We will show that a collection of $k + 1$ lines, no two of which are parallel, will also meet at a common point.

Consider the set of $k + 1$ distinct lines in the plane. By our hypothesis, the first k of these lines meet in a common point p_1. Similarly the last k of these lines meet in a common point p_2. We will show that p_1 and p_2 must be the same point.

If p_1 and p_2 were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, p_1 and p_2 are the same point. We conclude that the point $p_1 = p_2$ lies on all $k + 1$ lines, thus our $k + 1$ distinct lines meet at a common point.

2. The set of *full binary (and rooted) trees*, where a rooted tree consists of a set of vertices containing a distinguished vertex called the *root*, and eyes connecting these vertices, can be defined recursively by:

   Basis Step: A single vertex r is a full binary (and rooted) tree.
   Recursive Step: Suppose that T_1 and T_2 are disjoint full binary (and rooted) trees, there is a full binary (and rooted) tree, denoted $T_1 \cdot T_2$, consisting for a root r together with edges connecting r to the root of the left subtree T_1 and r to the root of the right subtree T_2.

   (a) Draw the basis step tree.

   (b) Draw the family of trees one step "above" the basis step tree.

   (c) Draw the family of trees two steps "above" the basis step tree.

   (d) Let $h(T)$ be the height of a tree.
   We define $h(\cdot) = 0$, and $h(T_1T_2) = 1 + \max\{h(T_1), h(T_2)\}$.
   Find the height of the trees in (c).

3. Let T be a binary tree, and $n(T)$ be the number of vertices in T. Use induction to prove if T is a full binary tree, then $n(T) \geq 2h(T) + 1$.