Many Different Algorithms (and why they work!)

Addition

- Concrete model (manipulatives)
  In order to be effective, this algorithm ***MUST* involve place value.

Example: $5492_{ten} + 1827_{ten}$. This problem could be modeled by place value blocks or chips. For example, a red chip would be equal to 1000 unit chips (white), a green chip would be equal to 100 unit chips (white), a blue chip would be equal to 10 unit chips (white), and a white chip would be worth 1 unit. Notice that a green chip could be decomposed into 100 unit chips (white) or ten (10) of the next lowest unit (blue).

$5492_{ten} + 1827_{ten} = (5R + 4G + 9B + 2W) + (1R + 8G + 2B + 7W) = (6R + 12G + 11B + 9W)$

At this point, the "re-grouping" or "composing" would be illustrated. Each time a higher unit can be composed, it is. The 11B would be broken down into 10B + 1B = 1G + 1B. The newly composed green chip is then placed with the other green chips. Similar compositions occur with the green chips. 13G = 10G + 3G = 1R + 3G. The final result is shown below:

$6R + 12G + 11B + 9W = 6R + 13G + 1B + 9W = 7R + 3G + 1B + 9W = 7319_{ten}$.

- Expanded with properties (note the similarity to the previous model).

$5492_{ten} + 1827_{ten} =$	Original addition statement
$(5\cdot1000+4\cdot100+9\cdot10+2\cdot1)+(1\cdot1000+8\cdot100+2\cdot10+7\cdot1)=$	Expanded form
$(5\cdot1000+1\cdot1000)+(4\cdot100+8\cdot100)+(9\cdot10+2\cdot10)+(2\cdot1+7\cdot1)=$	Associative & Commutative Properties
$6\cdot1000+12\cdot100+11\cdot10+9\cdot1=$	Addition of like terms
$6\cdot1000+(10\cdot100+2\cdot100)+(10\cdot10+1\cdot10)+9\cdot1=$	Decomposition
$6\cdot1000+(1\cdot1000+2\cdot100)+(1\cdot100+1\cdot10)+9\cdot1=$	Multiplication of powers of ten
$(6\cdot1000+1\cdot1000)+(2\cdot100+1\cdot100)+1\cdot10+9\cdot1=$	Associative Property of Addition
$(7\cdot1000)+(3\cdot100)+1\cdot10+9\cdot1=$	Addition of like terms
7319_{ten}	Writing in non - expanded form

Each step here must be justified mathematically with properties. This actually shows "why" addition in the base-ten place value system will work.

- Left-to-Right (possibly with cross-out)

```
  5492
+ 1827     → The answer would be 7319.
  6219
```

The idea here is to add quickly from left to right. Whenever the sum of the digits of the next place (to the right) is larger than 10, draw a line through the larger place value. When done, add one (1) to each of the crossed-out segments. Do you see why? Could you explain what to do with this?

- Traditional algorithm (the one that most of you grew up with… remember "carrying"?)

```
 1 1
 5492
+ 1827
 7319
```

The idea here is to write the "fair-trade" or "regrouped" amount at the top of the next place value to save time. Most of you would use this, but probably only because it was taught to you and drilled into your head over time. I strongly encourage you to try other methods and algorithms shown here.

- Partial sums (similar to the traditional algorithm)

```
  5492
+ 1827
     9
   110
  1200
+ 6000
  7319
```

The idea here is to show each of the additions that are performed. Since there are 4-digits (and therefore 4 place values), there will be 4 partial sums. Place value is key, and should be emphasized here. Do you see how you could use this model to show where the traditional algorithm comes from?

- Addition and subtraction (trading off using compatible numbers)

$$5492+1827=5492+1827+0=5492+1827+(8-8)=$$
$$(5492+8)+(1827-8)=5500+1819=7319$$

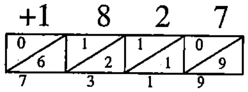
OR

$$5492+1827=5492+1827+0=5492+1827+(508-508)=$$
$$(5492+508)+(1827-508)=6000+1319=7319$$

The idea here is to make the addition easier by adding and subtracting the same number to the equation. This works best when the numbers are close to the next highest place value (notice how 5492 is really close to 5500). Remember, when adding numbers together, we can not add the same number to both addends and end up with the correct sum – we must add and subtract so the sum is unchanged.

- Lattices (these used to be the primary method for teaching addition and multiplication but drawing the lattices has become too tedious for many – it is still quite helpful)

```
  5   4   9   2
 +1   8   2   7
```



The lattice at the bottom represents the place values – but on diagonals. When adding single digits, the most you'd end up with is a number one place value greater. When you finish adding the single digits, you add up the numbers in each diagonal (these are the same place values)

- Scratch Addition (this method works best when adding large columns of numbers together)

495		$\overset{2}{4}95$		$\overset{2\,2}{4}\,9_1\,5$		$\overset{2\,2}{4}\,9_1\,5$
329		329_4		329_4		329_4
432	→	432	→	432	→	$4_3 32$
763		763		$76_2 3$		$7_0 6_2 3$
+837		$+837_6$		$+837_6$		$+837_6$
		6		56		2856

As you add down a place value column, if you encounter a sum greater than 10, put a scratch through that digit and jot down the remaining single units. Continue using only the single units until you have finished a place value. The number of scratches is the number of "tens" for that place value. You'll need to compose those tens into the next place value – so write the number of scratches (tens) in the next place value to the left. Repeat until finished. This makes addition FUN again!

Subtraction

- Concrete model (manipulatives)
  In order to be effective, this algorithm ***MUST* involve place value.

  Example: $542_{ten} - 187_{ten}$. This problem could be modeled by place value blocks or chips. For example, a green chip would be equal to 100 unit chips (white), a blue chip would be equal to 10 unit chips (white), and a white chip would be worth 1 unit. Notice that a green chip could be decomposed into 100 unit chips (white) or ten (10) of the next lowest unit (blue).

  $542_{ten} - 187_{ten} = (5G + 4B + 2W) - (1G + 8B + 7W)$

  At this point, the "re-grouping", "fair-trading", or "decomposing" would be illustrated. If a subtraction can not take place (physically), then decompose a unit from a higher place value to a lower value. The 4B would be broken down into 3B + 1B = 3B + 10W. The decomposed white chips are then placed with the other white chips. Similar decompositions occur with the green chips. 5G = 4G + 1G =4G + 10R. The final result is shown below:

  $(5G + 4B + 2W) - (1G + 8B + 7W) = (5G + 3B + 12W) - (1G + 8B + 7W) =$

  $(4G + 13B + 12W) - (1G + 8B + 7W) = 3G + 5B + 5W = 355_{ten}$.

Notice that the result is turned back into the correct base number. You could choose to compose and decompose numbers along the way, as shown below:

$(5G + 4B + 2W) - (1G + 8B + 7W) = (5G + 3B + 12W) - (1G + 8B + 7W) =$

$(5G + 3B + 5W) - (1G + 8B) = (4G + 13B + 5W) - (1G + 8B) = 3G + 5B + 5W = 355_{ten}$.

- Expanded with properties (note the similarity to the previous model).

$542_{ten} - 187_{ten} =$	Original subtraction statement
$(5\cdot 100 + 4\cdot 10 + 2\cdot 1) - (1\cdot 100 + 8\cdot 10 + 7\cdot 1) =$	Expanded form
$((4\cdot 100 + 1\cdot 100) + (3\cdot 10 + 1\cdot 10) + 2\cdot 1) - (1\cdot 100 + 8\cdot 10 + 7\cdot 1) =$	Rewriting the number using addition
$((4\cdot 100 + 10\cdot 10) + (3\cdot 10 + 10\cdot 1) + 2\cdot 1) - (1\cdot 100 + 8\cdot 10 + 7\cdot 1) =$	Decomposition into lower place values
$(4\cdot 100 + (10\cdot 10 + 3\cdot 10) + (10\cdot 1 + 2\cdot 1)) - (1\cdot 100 + 8\cdot 10 + 7\cdot 1) =$	Associative Property of Addition
$(4\cdot 100 + 13\cdot 10 + 12\cdot 1) - (1\cdot 100 + 8\cdot 10 + 7\cdot 1) =$	Addition
$4\cdot 100 + 13\cdot 10 + 12\cdot 1 - 1\cdot 100 - 8\cdot 10 - 7\cdot 1 =$	Distributive Property of Mult. over Add.
$4\cdot 100 - 1\cdot 100 + 13\cdot 10 - 8\cdot 10 + 12\cdot 1 - 7\cdot 1 =$	Commutative Property of Addition
$3\cdot 100 + 5\cdot 10 + 5\cdot 1 =$	Subtraction
355_{ten}	Writing in non - expanded form

Each step here must be justified mathematically with properties. This actually shows "why" subtraction in the base-ten place value system will work.

- Regrouping in multiple ways (similar to what you read in Liping Ma, Chapter 1)

$$542 - 187 = (342 + 200) - 187 = 342 + (200 - 187) = 342 + 13 = 355$$

OR

$$542 - 187 = (442 + 100) - (100 + 87) = 442 - 100 + (100 - 87) = 342 + 13 = 355$$

OR

$$542 - 187 = (432 + 100 + 10) - (100 + 80 + 7) = (432 - 100) + (100 - 80) + (10 - 7) = 332 + 20 + 3 = 355$$

OR

$$542 - 187 = (400 + 130 + 12) - (100 + 80 + 7) = (400 - 100) + (130 - 80) + (12 - 7) = 300 + 50 + 5 = 355$$

The last one very closely resembles the concrete model.

- Equal addends (add [or subtract] the same number to the subtrahend and minuend, the difference will not change)

$$542 - 187 = (542 + 8) - (187 + 8) = (550) - (195) = 355$$

OR

$$542 - 187 = (542 + 13) - (187 + 13) = (555) - (200) = 355$$

OR

$$\begin{array}{r} 542 \\ \underline{-187} \\ \end{array} \rightarrow \begin{array}{r} 54{}^{1}2 \\ \underline{-1\overset{9}{8}7} \\ 5 \end{array} \rightarrow \begin{array}{r} 54{}^{1}2 \\ \underline{-\overset{2}{1}\overset{9}{8}7} \\ 55 \end{array} \rightarrow \begin{array}{r} 5{}^{1}4{}^{1}2 \\ \underline{-\overset{2}{1}\overset{9}{8}7} \\ 355 \end{array}$$

In the last example shown, ten units are added to the subtrahend and minuend. In the minuend (542), we add the ten units as just units. In the subtrahend (187), we add the ten units as one group of ten. To finish, add one hundred units to each – ten groups of ten in the minuend, and one group of one hundred in the subtrahend. This is the most common algorithm in the world… but you are probably more used to the next one.

- Traditional algorithm (fair-trading and regrouping with minuend only)

$$\begin{array}{r} 542 \\ \underline{-187} \\ \end{array} \rightarrow \begin{array}{r} {\scriptstyle 3\,1} \\ 5\not{4}\not{2} \\ \underline{-187} \\ 5 \end{array} \rightarrow \begin{array}{r} {\scriptstyle 13\,1} \\ \not{5}\not{4}\not{2} \\ \underline{-187} \\ 55 \end{array} \rightarrow \begin{array}{r} {\scriptstyle 4\;13\,1} \\ \not{5}\not{4}\not{2} \\ \underline{-187} \\ 355 \end{array}$$

There is nothing at all wrong with this algorithm. Make sure that students understand the decomposition that is taking place. Each step along the way, the minuend is STILL 542 units. However, you really end up breaking it into place values that look quite strange. Really you are re-writing 542 as 41312... where there are 4 hundreds, 13 tens, and 12 ones. There is nothing wrong with this idea, but we wouldn't want to write the number in this form (remember the place value of each digit is given – not a group of two digits)

- Partial differences (only works when students have seen negative numbers, but ooooh...it is slick! This can be modeled by the chips as well, where negative numbers are like the amount owed.)

$$\begin{array}{r} 542 \\ \underline{-187} \end{array} \rightarrow \begin{array}{r} 542 \\ \underline{-187} \\ -5 \end{array} \rightarrow \begin{array}{r} 542 \\ \underline{-187} \\ -5 \\ -40 \end{array} \rightarrow \begin{array}{r} 542 \\ \underline{-187} \\ -5 \\ -40 \\ \underline{400} \\ 355 \end{array}$$

The idea here is that you can take 7 from 2... however, you end up with a negative number. Complete the subtraction at the end – if using chips, the negative amounts would cause a decomposition of the hundreds into smaller amounts.

You could look at this algorithm as the following (using the distributive property):

$542-187=$	Original subtraction statement
$(5\cdot100+4\cdot10+2\cdot1)-(1\cdot100+8\cdot10+7\cdot1)=$	Expanded form
$5\cdot100+4\cdot10+2\cdot1-1\cdot100-8\cdot10-7\cdot1=$	Distributive Property of Mult. over Add.
$(5\cdot100-1\cdot100)+(4\cdot10-8\cdot10)+(2\cdot1-7\cdot1)=$	Associative Property of Addition
$4\cdot100-4\cdot10-5\cdot1=$	Subtraction
$(3\cdot100+1\cdot100)-4\cdot10-5\cdot1=$	Rewriting the hundreds into a sum using addition
$(3\cdot100+10\cdot10)-4\cdot10-5\cdot1=$	Decomposing hundreds into tens
$(3\cdot100+9\cdot10+10\cdot1)-4\cdot10-5\cdot1=$	Rewriting the tens into a sum using addition
$3\cdot100+5\cdot10+5\cdot1=$	Subtraction
355	Writing in non - expanded form

- Subtract from the base (very quick after you get the hang of it)

$$\begin{array}{r} 542 \\ \underline{-187} \\ \end{array} \quad \rightarrow \quad \begin{array}{r} 5\overset{3}{4}\overset{10}{2} \\ \underline{-187} \\ 5 \end{array} \quad \rightarrow \quad \begin{array}{r} \overset{4}{5}\overset{\overset{10}{3}}{4}\overset{10}{2} \\ \underline{-187} \\ 55 \end{array} \quad \rightarrow \quad \begin{array}{r} \overset{4}{5}\overset{\overset{10}{3}}{4}\overset{10}{2} \\ \underline{-187} \\ 355 \end{array}$$

While very much like our traditional algorithm, the idea behind it is much different. The first step is to decompose one 10 into ten 1s. Then subtract the 7 ones from the 10 ones… leaving 3 ones. Add this to the original two ones and you're on your way! Repeat for all place values where the face value in the subtrahend is greater than the face value in the minuend. Why is this one better sometimes? This requires only knowledge of the number 10 minus a single digit. Students struggling with subtraction of many numbers, 17 – 9 for example, would have an alternative. Of course we want students to be more proficient with those subtractions, but it should not hold them back!

- Add the complement (use a fancy re-writing to turn subtraction into addition!)

$$\begin{array}{r} 542 \\ \underline{-187} \\ \end{array} \quad \rightarrow \quad \begin{array}{r} 542 \\ \underline{+813} \\ \end{array} \quad \rightarrow \quad \begin{array}{r} 542 \\ \underline{+813} \\ 1355 \end{array} \quad \rightarrow \quad \begin{array}{r} 542 \\ \underline{-187} \\ 355 \end{array}$$

Here's the idea behind this gem! 1000 – 187 = 813. Therefore, 542 + 813 is the same thing as $542+813=542+(1000-187)=1000+(542-187)$. *So, add the "complement" to 542 and then subtract 1000. The "complement" in this case is referring to the number that would sum to be "one" of the next highest place value. The "complement" of 333 would be 667. The complement of 246 would be 754. Once you get the hang of this, it is quite easy and rather ingenious.*

Multiplication

- Partial Products.

$$\begin{array}{r} 19 \\ \underline{\times 37} \end{array} \quad \rightarrow \quad \begin{array}{rl} 19 & \\ \underline{\times\ 37} & \\ 63 & (9\times 7) \\ 70 & (10\times 7) \\ 270 & (30\times 9) \\ \underline{+\ 300} & \underline{(10\times 30)} \\ 703 & \end{array}$$

Each place value is multiplied by each other place value, results are written individually and then summed. Note the similarity to the standard algorithm you probably learned.

- Standard algorithm.

$$
\begin{array}{r} 19 \\ \underline{\times 37} \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 6 \\ 19 \\ \underline{\times 37} \\ 3 \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 6 \\ 19 \\ \underline{\times 37} \\ 133 \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 2 \\ \scriptstyle \not 6 \\ 19 \\ \underline{\times 37} \\ 133 \\ 70 \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 2 \\ \scriptstyle \not 6 \\ 19 \\ \underline{\times 37} \\ 133 \\ \underline{570} \\ 703 \end{array}
$$

You all probably grew up with this one. Notice the regrouping as you go along. It is important to understand WHY there is a "0" to the right of the 7 in the second partial product. You should also understand what the digits written above the other numbers mean.

- Lightning method (sometimes called the cross product method).

$$
\begin{array}{r} 19 \\ \underline{\times 37} \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 6 \\ 19 \\ \underline{\times 37} \\ 3 \end{array}
\rightarrow
\begin{array}{r} \scriptstyle 4 \\ 19 \\ \underline{\times 37} \\ 03 \end{array}
\rightarrow
\begin{array}{r} 19 \\ \underline{\times 37} \\ 703 \end{array}
$$

This algorithm is most often done without writing things down. I have written numbers above to show you what you would be storing mentally. Starting with the ones, you multiply and write down the ones digit of the ones (there can be no other ones formed in this product). Then, multiply the ones unit by the tens unit of each factor – when done, add these to any groups of ten from the product of the units. Write down the "units" digit of this product, as it will be the number of tens (there can be no other tens formed in this product). Lastly, multiply the tens digit by the tens digit of the other number and add on any remaining groups of hundred.

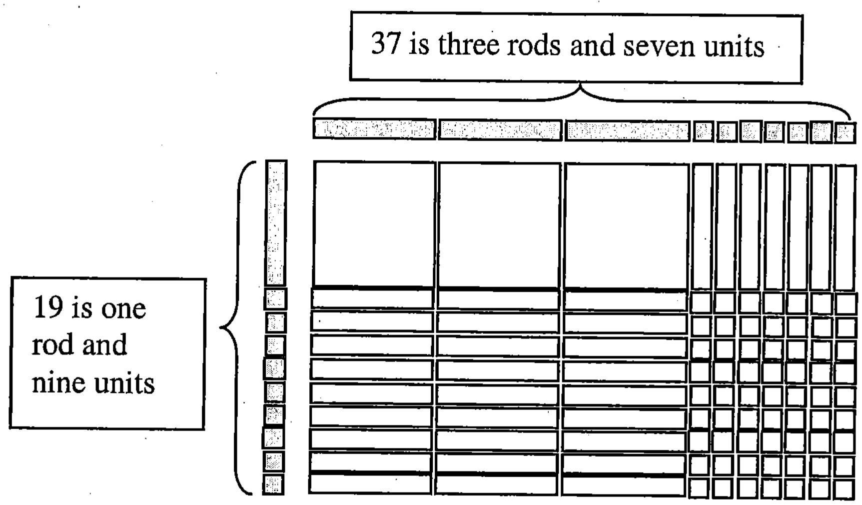
*Mentally, it would read as follows: (1) multiply 9*7 = 63, write down the 3 and remember the 6 (groups of ten); (2) multiply 1*7 and 3*9 to get 7 and 27, which is 34... but I also add the 6 groups of ten I had before to get 40 (groups of ten) – write down the 0 and remember the 4 (groups of hundred); (3) multiply 1*3 to get 3 and add the 4 from before to end with 7 hundreds.*

- Slow-lightning method

$$
\begin{array}{r} 19 \\ \underline{\times 37} \end{array}
\rightarrow
\begin{array}{r} 19 \\ \underline{\times 37} \\ 363 \end{array}
\rightarrow
\begin{array}{r} 19 \\ \underline{\times 37} \\ 363 \\ \underline{340} \\ 703 \end{array}
$$

This algorithm is a slow version of the lightning algorithm. Here, you multiply the ones units and write all of them down, then multiply the hundreds units and write them on the same line. Since there are only two digits, there will never be any overlap! Lastly, multiply the ones times tens for each number and add together. Since these will be groups of tens, place them in the appropriate space. Add the result! Once perfected, this method is far quicker than the standard algorithm. It CAN be modified for larger numbers with some thought. Can you see how?

- Concrete model – area using base blocks. Base blocks are blocks broken into the different place value units. There are "units" (worth 1), "rods" (worth the base amount), "flats" (worth the base amount squared), and "cubes" (worth the base amount cubed). 19×37 is shown below:



The shaded (yellow or grey depending on the printout) regions to the top and left of the rectangular area are the guides… remove them to calculate the area. In base ten, each rod is one ten and each flat is one hundred. Notice that there are 3 flats to begin with, 27 rods in one group and 7 in another, and 63 units. This would appear as 3F, 34R, 63U. The units can be composed into groups of 10… six of these. Therefore, we'd see that 3F +34R + 63U = 3F + 40R + 3U = 7F + 0R + 3U = 703.

- Expanded with properties

$37 \cdot 19 =$	Original multiplication problem
$(30+7)\cdot(10+9)=$	expanded form
$30\cdot10+30\cdot9+7\cdot10+9\cdot7=$	distributive property of mult. over add.
$(3\cdot10)\cdot10+(3\cdot10)\cdot9+7\cdot10+9\cdot7=$	factoring into powers of ten
$3\cdot(10\cdot10)+(3\cdot9)\cdot10+7\cdot10+9\cdot7=$	associative and commutative properties of mult.
$3\cdot(10\cdot10)+(3\cdot9+7)\cdot10+9\cdot7=$	factoring (reverse distributive prop. of m. over add.)
$3\cdot100+34\cdot10+63=$	multiplication and addition
$3\cdot100+(30+4)\cdot10+(60+3)=$	expanded form of new numbers
$3\cdot100+(30\cdot10)+(4\cdot10)+(6\cdot10+3)=$	distributive property of mult. over add.
$3\cdot100+3\cdot(10\cdot10)+(4+6)\cdot10+3=$	factoring and associative prop. of mult.
$3\cdot100+3\cdot100+10\cdot10+3=$	multiplication and addition
$3\cdot100+3\cdot100+1\cdot100+3=$	multiplication
$(3+3+1)\cdot100+3=$	factoring (reverse distributive prop.)
$7\cdot100+3=$	addition
703	writing in non - expanded form

- Multiplication by 10 (or by 10^n)
  $37\cdot 19=(37)\cdot(10+9)=37\cdot 10+37\cdot 9=370+333=703$

  Multiplication by 10 is a great tool. Any number multiplied by 10^n simply moves the place value of each digit "n" place value units. Below is another example... and nicely enough, this works perfectly in EVERY number base.

  $$6437\cdot 100=(6\cdot 10^3+4\cdot 10^2+3\cdot 10^1+7)\cdot(10^2)=$$
  $$6\cdot 10^3\cdot(10^2)+4\cdot 10^2\cdot(10^2)+3\cdot 10^1\cdot(10^2)+7\cdot(10^2)=6\cdot 10^5+4\cdot 10^4+3\cdot 10^3+7\cdot 10^2=643{,}700$$

- Concrete model 2 (with the chips and the multiplication by 10)
  In order to be effective, this algorithm ***MUST* involve place value.
  Example: $37_{ten}\cdot 19_{ten}$. This problem could be modeled by place value blocks or chips. For example, a red chip would be equal to 1000 unit chips (white), a green chip would be equal to 100 unit chips (white), a blue chip would be equal to 10 unit chips (white), and a white chip would be worth 1 unit. Notice that a green chip could be decomposed into 100 unit chips (white) or ten (10) of the next lowest unit (blue).

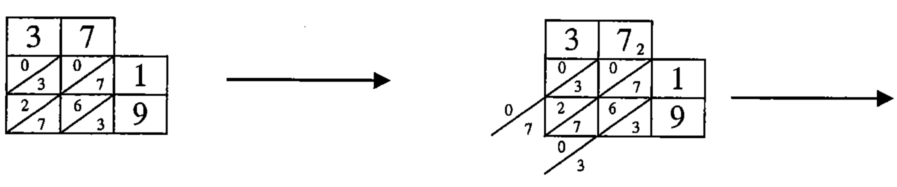
  $$37_{ten}\cdot 19_{ten}=37\cdot(10+9)=37\cdot 10+37\cdot 9=37\cdot 10+(30+7)\cdot 9=37\cdot 10+30\cdot 9+7\cdot 9$$

  You could choose to make 37_{ten} piles of 19_{ten}, or even 19_{ten} piles of 37_{ten}. However, by using the multiplication by 10, the problem breaks down much easier. To model this, notice that $37\cdot 10=(3B+7G)\cdot 10=3G+7B$... the place values move by one unit. Next, $30\cdot 9=(3B)\cdot 9=27B$ and $7W\cdot 9W=63W$. The entire problem has been broken down into the pieces.

  $$37_{ten}\cdot 19_{ten}=37\cdot 10+30\cdot 9+7\cdot 9=(3G+7B)+27B+63W=3G+34B+63W$$
  At this point, the "re-grouping" or "composing" would be illustrated. Each time a higher unit can be composed, it is. The 11B would be broken down into 10B + 1B = 1G + 1B. The newly composed green chip is then placed with the other green chips. Similar compositions occur with the green chips. 13G = 10G + 3G = 1R + 3G. The final result is shown below:

  $$37_{ten}\cdot 19_{ten}=3G+34B+63W=3G+40B+3W=7G+0B+3W=703_{ten}$$

- Lattice multiplication ☺



0703 = 703

This method is fantastic because it reduces multi-digit multiplication to single digit multiplication. These single-digit products are written with proper units for the given number base. The diagonal lines represent the place values...all possibilities are covered. Tens could be obtained from a large product of the ones digits, or from the product of the tens and ones. You should be able to tell the place value from looking at the chart as well as understanding place value of the product.

- Doubling (with base 2!)

$$\begin{array}{lcll}
37\cdot 1=37 & & 37\cdot 1=37\ \text{x} & \\
37\cdot 2=74 & & 37\cdot 2=74\ \text{x} & 592 \\
37\cdot 4=148 & \rightarrow & 37\cdot 4=148 & \rightarrow\ 37 \\
37\cdot 8=296 & & 37\cdot 8=296 & +74 \\
37\cdot 16=592 & & 37\cdot 16=592\ \text{x} & 703
\end{array}$$

In this algorithm, you merely take one number and double it repeatedly. Stop doubling when you reach a scale-factor larger than your other factor from the original product. In this case, $37\cdot 19$ *is the original problem. Doubling to 16 is quick, and we would not need to go to 32 (double again) since this is larger than 19. Now just take the new products that will add up to* $37\cdot 19$. *In this case, 19 = 16 + 2 + 1. To find the product, add the rows that have an "x" next to them. You could do this with either number to start as shown below… so choose wisely. Notice that 37 = 32 + 4 + 1.*

$$\begin{array}{lcll}
19\cdot 1=19 & & 19\cdot 1=19\ \text{x} & \\
19\cdot 2=38 & & 19\cdot 2=38 & 608 \\
19\cdot 4=76 & & 19\cdot 4=76\ \text{x} & 76 \\
19\cdot 8=152 & \rightarrow & 19\cdot 8=152 & \rightarrow\ +19 \\
19\cdot 16=304 & & 19\cdot 16=304 & 703 \\
19\cdot 32=608 & & 19\cdot 32=608\ \text{x} &
\end{array}$$

- Russian Peasant Method (utilizes doubling AND halving)

19	37
9	74
4	148
2	296
1	592

→

19	37
9	74
~~4~~	~~148~~
~~2~~	~~296~~
1	592

→

37
74
+592
703

37	19
18	38
9	76
4	152
2	304
1	608

→

37	19
~~18~~	~~38~~
9	76
~~4~~	~~152~~
~~2~~	~~304~~
1	608

→

19
76
+608
703

In this algorithm, one number is doubled and the other halved. If you have an odd number, drop the remainder – and this is key. When the halving column reaches "1", you are done. Now, cross out each row that has an even number in the halving column. Add up the remaining entries in the doubling column.

So, why does this work? It is really identical to the previous one, and utilizes converting to base 2. To convert any base ten number to any other number base, you can just take the number and divide repeatedly by the base number. Write down only the remainders at each stage… these form the numeral in the new base. When working with base 2, the remainder is either 1 or 0. If the remainder is 0, then the digit is removed from consideration in the final term.

Division

- Concrete methods

  In order to be effective, this algorithm ***MUST* involve place value.

  Example: $4573_{ten} \div 6_{ten}$. This problem could be modeled by place value blocks or chips. For example, a red chip would be equal to 1000 unit chips (white), a green chip would be equal to 100 unit chips (white), a blue chip would be equal to 10 unit chips (white), and a white chip would be worth 1 unit. Notice that a green chip could be decomposed into 100 unit chips (white) or ten (10) of the next lowest unit (blue). In this case, consider making "6_{ten}" equal piles – partition method of division.

$4573_{ten} \div 6_{ten} = (4R + 5G + 7B + 3W) \div 6$

Using regrouping or decomposition, my goal is to make piles that are divisible by 6. If you have the chips in front of you, you can distribute the chips into 6 piles as you go.

$4R + 5G + 7B + 3W = 0R + 45G + 7B + 3W = 0R + 42G + 37R + 3W =$

$0R + 42G + 36R + 13W = (0R + 42G + 36R + 12W) + 1W$

The result (in parenthesis) can be easily divided into 6 equal groups. Each group would have $0R + 7G + 6B + 2W = 762_{ten}$, with a remainder of one white chip.

$4573_{ten} \div 6_{ten} = 762_{ten} \quad R1$

You could also do larger problems with the chips – even involving place value.
Example: $4573_{ten} \div 34_{ten}$.

We can think of making "34" groups… but that may be a little much! It would work, but there is a better way. Remember multiplication by 10 moves the place values by one. Therefore, we could look for groups of "34", "340", or even "3400.

$4573_{ten} = 4R + 5G + 7B + 3W = (3R + 4G) + 1R + 1G + 7B + 3W = (3R + 4G) + 0R + 11G + 7B + 3W =$
$(3R + 4G) + 0R + 10G + 17B + 3W = (3R + 4G) + (3G + 4B) + (3G + 4B) + (3G + 4B) + 1G + 5B + 3W =$
$(3R + 4G) + [(3G + 4B) + (3G + 4B) + (3G + 4B)] + 0G + 13B + 23W =$
$(3R + 4G) + [(3G + 4B) + (3G + 4B) + (3G + 4B)] + 13B + 23W =$
$(3R + 4G) + [(3G + 4B) + (3G + 4B) + (3G + 4B)] + 4 \cdot [(3B + 4W)] + 1B + 7W =$
$1 \cdot (3R + 4G) + 3 \cdot (3G + 4B) + 4 \cdot (3B + 4W) + 1B + 7W =$
$1 \cdot (3400) + 3 \cdot (340) + 4 \cdot (34) + 17$
Therefore, $4573_{ten} \div 34_{ten} = 134 \quad R17$

- Standard

$4573_{ten} \div 34_{ten}$

$$\begin{array}{r} 1 \\ 34\overline{)4573} \\ -\underline{34}\downarrow \\ 117 \end{array} \rightarrow \begin{array}{r} 13 \\ 34\overline{)4573} \\ -\underline{34} \\ 117 \\ -\underline{102}\downarrow \\ 153 \end{array} \rightarrow \begin{array}{r} 134 \\ 34\overline{)4573} \\ -\underline{34} \\ 117 \\ -\underline{102} \\ 153 \\ -\underline{136} \\ 17 \end{array} \rightarrow 4573_{ten} \div 34_{ten} = 134 \;\; R17$$

The entire basis of this algorithm is place value. At first, we are not asking how many "34's" are in "45", we are asking how many 34's are in 4500. A clearer way to write this, which shows all of the partial steps and doesn't lose out on the idea of place value is as follows:

- Scaffolding

$$\begin{array}{r} 100 \\ 34\overline{)4573} \\ -\underline{3400} \\ 1170 \end{array} \rightarrow \begin{array}{r} 30 \\ 100 \\ 34\overline{)4573} \\ -\underline{3400} \\ 1170 \\ -\underline{1020} \\ 153 \end{array} \rightarrow \begin{array}{r} 4 \\ 30 \\ 100 \\ 34\overline{)4573} \\ -\underline{3400} \\ 1170 \\ -\underline{1020} \\ 153 \\ -\underline{136} \\ 17 \end{array} \rightarrow 4573_{ten} \div 34_{ten} = 134 \;\; R17$$

Build up on your quotient, modifying it as you go. When finished, add up all components... voila! This idea works great for any number base, and is extra nice because you don't have to have the exact quotient as you go. If you know there are "2" of them, but aren't sure about "3", use "2" and then subtract. If your remaining portion is larger than your divisor, you just subtract another one. When finished, add up all your mini-quotients to find the final quotient.

- Short-division

$$4573_{ten} \div 6_{ten} \rightarrow \begin{array}{r} 7 \\ 6\overline{)4\,5\,{}_{3}7\,3} \end{array} \rightarrow \begin{array}{r} 7\,6 \\ 6\overline{)4\,5\,{}_{3}7\,{}_{1}3} \end{array} \rightarrow \begin{array}{r} 7\,6\,2 \\ 6\overline{)4\,5\,{}_{3}7\,{}_{1}3\,{}_{1}} \end{array}$$

$4573_{ten} \div 6_{ten} = 762_{ten} \;\; R1$

To save time, the division is performed mentally with the quotient and remainder written down. The remainder is written smaller in front of the next place value, and the process continues.