Key

Final | TMath 126 | Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and will likely have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) If $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$ and $\vec{a} \times \vec{b} = 0$, then $\vec{a}$ is perpendicular to $\vec{b}$.

   False. recall $\|\vec{a}\times\vec{b}\|$ = area of parallelogram created by $\vec{a}$ & $\vec{b}$
   If $\|\vec{a}\times\vec{b}\|=0$ then the parallelogram has 0 area $\Rightarrow \vec{a} /\!/ \vec{b}$
   OR $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta = 0 \Rightarrow \theta = 0°$ so $/\!/$

   (b) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges to a finite number.

   False! The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$
   is such that $\lim_{n\to\infty} \frac{1}{n} = 0$ but the series
   does NOT converge to a finite number.

   (c) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that the n^{th} partial sum of a series is $s_n = \dfrac{n+5n^2}{n^2-e}$. Then $\lim_{n\to\infty} a_n = 5$.

   False. Note $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ where s_n is the n^{th} partial sum

   $$= \lim_{n\to\infty} \frac{n+5n^2}{n^2-e} \overset{L'H}{=} \lim_{n\to\infty} \frac{1+10n}{2n-0}$$

   $$\overset{L'H}{=} \lim_{n\to\infty} \frac{10}{2} = 5$$

   The series converges so the terms (a_n) converge to zero

(d) $||\vec{a} \times \vec{b}|| = ||\vec{b} \times \vec{a}||$. True

$||\vec{a} \times \vec{b}||$ is the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.
Since a parallelogram with sides $\vec{a}$ and $\vec{b}$ has the same area as a parallelogram with sides $\vec{b}$ and $\vec{a}$, the 2 expressions are equal.

(e) $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$. True

$\vec{a} \times \vec{b}$ results in a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$.
So $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ → the angle is $90°$.
Since $\vec{x} \cdot \vec{y} = ||\vec{x}|| \, ||\vec{y}|| \cos\theta$ where θ is the angle between $\vec{x}$ & $\vec{y}$
we know $(\vec{a} \times \vec{b}) \cdot \vec{a} = ||\vec{a} \times \vec{b}|| \, ||\vec{a}|| \cos 90° = 0$.

(f) Let f be a function of x and y. If $\nabla f(c,d) = (2,1)$, then the vector $\langle 2,1 \rangle$ is tangent to the contour line of the surface of f at $(c,d,f(c,d))$.

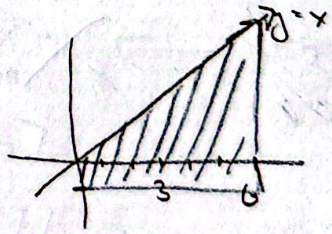
False. $\nabla f(c,d)$ points in the direction of the steepest ascent.
The contour line would keep the "elevation" constant

(g) $\int_{-1}^{2} \int_{0}^{6} x^2 \sin(x-y)\,dx\,dy = \int_{0}^{6} \int_{-1}^{2} x^2 \sin(x-y)\,dy\,dx$

True. The function $x^2\sin(x-y)$ is continuous on the rectangle $[0,6]\times[-1,2]$ so we can use Fubini's Thm.

(h) $\int_{-1}^{x} \int_{0}^{6} x^2 \sin(x-y)\,dx\,dy = \int_{0}^{6} \int_{-1}^{x} x^2 \sin(x-y)\,dy\,dx$

False. The integral on the right is integrating over the shaded region. →



If we reversed the order of integration we'd see

$$\int_{-1}^{6} \int_{y}^{6} x^2 \sin(x-y)\,dx\,dy$$

2. Evaluate the following if possible.

$$\lim_{n\to\infty} a_n$$

where $a_1 = 0$ and $a_{n+1} = 2^{a_n} - 3$

Numerical method !!

n	a_n
1	0
2	$2^0 - 3 = -2.75$
3	$2^{-2.75} - 3 \approx -2.85$
4	-2.8614
5	-2.86239
6	-2.8624

looks to be about -2.86

$$\sum_{n=0}^{\infty} \frac{n+1}{3n+2}$$

algebraic theorem

Notice $a_n = \frac{n+1}{3n+2}$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{3n+2} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{3} = 1/3$$

Since $\lim_{n\to\infty} a_n \neq 0$ we know the series will diverge.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$$

$$\frac{1}{4} + \frac{-3}{4^2} + \frac{(-3)^2}{4^3} + \frac{(-3)^3}{4^4} + \dots$$

Geometric Series ?

Converges to $\frac{\text{first term}}{1 - \text{ratio}}$

$$\Rightarrow \frac{1/4}{1-(-3/4)} = \frac{1/4}{7/4} = \frac{1}{7}$$

$$\lim_{n\to\infty} \sin\left(\frac{6n\pi}{5+8n}\right)$$

This just looks like Math 124 :)

Since sine is continuous

$$\lim_{n\to\infty} \sin\left(\frac{6n\pi}{5+8n}\right) = \sin\left(\lim_{n\to\infty} \frac{6n\pi}{5+8n}\right)$$

$$\overset{L'H}{=} \sin\left(\lim_{n\to\infty} \frac{6\pi}{8}\right)$$

$$= \sin\left(6\pi/8\right) = \sin\left(3\pi/4\right)$$

$$= \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

3. The temperature of a microprocessor is taken every second and only the last three readings are recorded. Below is a chart of the temperature C (in Celsius) and time t from which we estimated the first and second derivatives of C at $t = 3$.

t	2	3	4
$C(t)$	46	48	52

n	0	1	2
$C^n(3) \approx$	48	4	3

We'll center it at 3 b/c

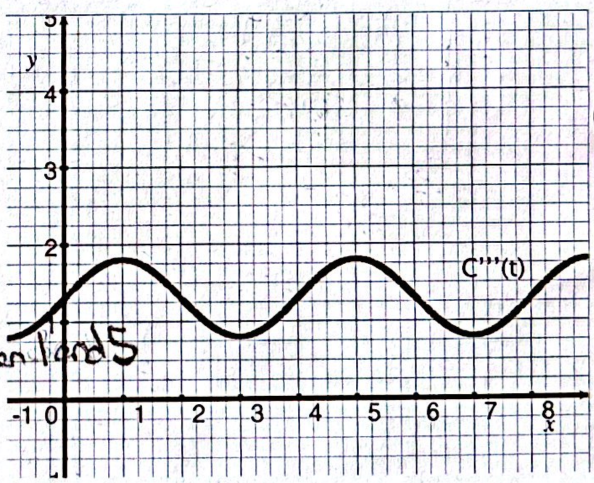
(a) Use all of the above data to estimate the values of C close to 3.

We can use a 2nd degree Taylor Approximation?

$C(x) \approx C(3) + \frac{1}{1!}(x-3)^1 C'(3) + \frac{1}{2!}(x-3)^2 C''(3)$

$C(x) \approx 48 + 4(x-3) + 3/2\,(x-3)^2$

(b) Temperature changes rather slowly and experimentally we know $C^{(3)}(t)$ has the following graph. Provide an upper bound for the estimate of $C(5)$ using part (a).



using (a) $C(5) \approx 48 + 4(5-3) + 3/2(5-3)^2 = 62$

Error $\leq \frac{1}{3!}(x-3)^3 \max|C^{(3)}(z)|$ where z is between 1 and 5

max value @ x=5 w/ value < 2

$\Rightarrow$ Error $\leq \frac{1}{3!}(5-3)^3 \cdot 2 = 2.67$

4. You are given the following data of a function $g(x, y)$. Your boss wants you to approximate $g(.8, 1.4)$ and wants to be convinced you're doing something sophisticated. Find a linear approximation for your boss and explain your choices (there are many that you will make!).

x	y	$g(x,y)$
0.55	1.2	27
0.65	1.0	31
0.65	1.1	29
0.75	1.2	50

linear approximation in 3D means find the tangent plane

We'll need $m_x = \frac{\partial g}{\partial x}\Big|_{point}$ and $m_y = \frac{\partial g}{\partial y}\Big|_{point}$

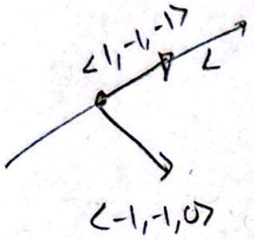
From the data given we can find $m_x = \frac{\partial g}{\partial x}\Big|_{y=1.2} \approx \frac{\Delta g}{\Delta x} = \frac{50-27}{.75-.55} = \frac{23}{.2} = 115$

and we can find $m_y = \frac{\partial g}{\partial y}\Big|_{x=.65}$ (between 2 computed) $\approx \frac{\Delta g}{\Delta y} = \frac{31-29}{1-1.1} = \frac{2}{-.1} = -20$

linear approx: use point $(.65, 1.1, 29)$

$z - 29 = 115(x-.65) - 20(y-1.1)$ so plug in $.8 = x$ and $y = 1.4 \Rightarrow z \approx 41.4$

5. Let Q be the plane containing the line $L(t) = \langle 2+t, 1-t, 1-t\rangle$ and the point $(1,0,1)$. Let R be defined by $x + 2y + 3z = 0$



(a) Find an equation of a plane for Q.

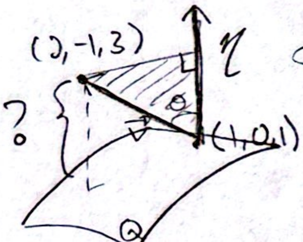
Note $L(0) = (2,1,1)$ so vector from $L(0)$ to $(1,0,1)$ is $\langle -1,-1,0\rangle$

The directional vector of $L(t)$ is $\langle 1,-1,-1\rangle$

$$\begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ -1 & -1 & 0 \end{vmatrix} = \vec{i}\,|0-1| - \vec{j}\,|0-1| + \vec{k}\,|-1-1| = \langle -1,1,-2\rangle$$

So $0 = \langle -1,1,-2\rangle \cdot (\langle x,y,z\rangle - (1,0,1))$ works

(b) Find the distance between Q and the point (2,-1,3).



So $\vec{\eta} = \langle -1,1,-2\rangle$ is normal to Q

Using that $(1,0,1)$ is on Q we can consider the vector $\vec{v}$ (graphed on left)

$\vec{v} = \langle 2-1, -1-0, 3-1\rangle = \langle 1,-1,2\rangle$

The distance is the "vertical" component drawn on the left.

We can use the dot product to find the angle of the Δ drawn & then find ?

Recall $\|\vec{v}\| \cdot \|\vec{\eta}\| \cos\theta = \vec{\eta} \cdot \vec{v}$

$\Rightarrow \cos\theta = \dfrac{-1-1-4}{\sqrt{1+1+4}\,\sqrt{1+1+4}}$

$\Rightarrow \cos\theta = \dfrac{-6}{6} = (-1)$

Using the shaded Δ

Sohcahtoa

$\cos\theta = ?/\|\vec{v}\|$

$\Rightarrow ? = \|\vec{v}\| \cos\theta$

subbing in what we know $\cos\theta$

$? = \|\vec{v}\|(-1)$

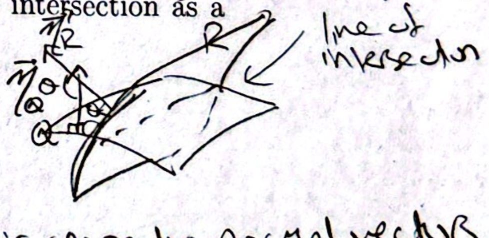
$? = \sqrt{1+1+4} = -\sqrt{6}$

So dist is $\sqrt{6}$

(c) Identify if R is a point, line, plane, or none of the above.

a plane Note R is equivalent to $\langle 1,2,3\rangle \cdot (\langle x,y,z\rangle - \langle 0,0,0\rangle) = 0$

(d) Given that Q and R intersect and identify the intersection as a point, line, plane, or none of the above.



(e) Find the angle that Q and R intersect.

The angle of intersection between Q & R will be the same angle as that between their respective normal vectors denoted $\vec{\eta}_Q$ & $\vec{\eta}_R$ respectively.

Recall $\vec{\eta}_Q \cdot \vec{\eta}_R = \|\vec{\eta}_Q\| \cdot \|\vec{\eta}_R\| \cos\theta$ where θ is the angle between $\vec{\eta}$

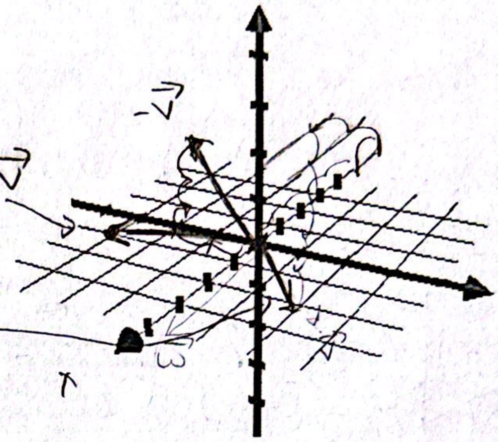
$\Rightarrow \langle -1,1,-2\rangle \cdot \langle 1,2,3\rangle = \sqrt{1+1+4}\,\sqrt{1+4+9}\,\cos\theta$

$\Rightarrow \dfrac{-1+2-6}{\sqrt{6}\,\sqrt{14}} = \cos\theta$

$\Rightarrow \dfrac{-5}{2\sqrt{3}\sqrt{7}} = \cos\theta$

$\theta = \arccos\left(\dfrac{-5}{2\sqrt{21}}\right)$

6. Consider the vectors: $\vec{v} = \langle 1, 2, -2\rangle$ and $\vec{w} = \langle 2, -1, -2\rangle$



(a) Draw the vector $-\vec{v}$

$= \langle -1, -2, 2\rangle$

(b) Draw the vector $\vec{w} - \vec{v}$

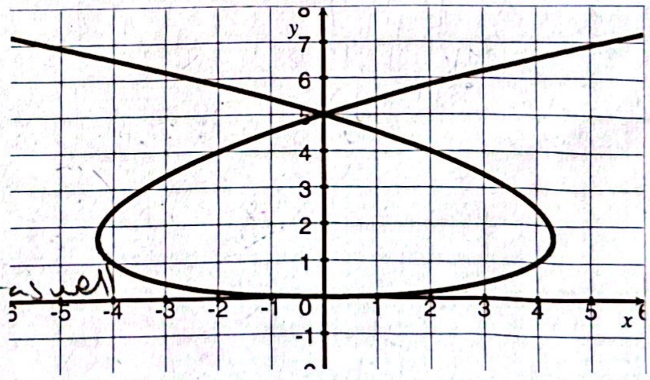
$= \langle 2,-1,-2\rangle - \langle 1,2,-2\rangle = \langle 1,-3,0\rangle$

(c) Draw the vector $\vec{v} \times \vec{w}$

(points into the paper)

$$\begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 2 & -1 & -2 \end{vmatrix} = \vec{i}(-4-2) - \vec{j}(-2+4) + \vec{k}(-1-4) = -6\vec{i} - 2\vec{j} - 5\vec{k}$$

7. Consider the parametric equation $x(t) = t^3 - 5t$ and $y(t) = t^2$.



(a) [3] Looking at the graph, approximate where $\frac{dy}{dx}$ is not defined.

at cusps or where there is a vertical tangent line so
$\approx (-4.4, 1.6)$ and $(4.4, 1.6)$
although the intersection is arguably a point as well

(b) [4] Find the equation of *one* of the lines tangent to the above parametric equations at $(0, 5)$.

Looking for $y = mx + b$
m = slope of line tangent to graph @ $(0,5)$
$= \left.\frac{dy}{dx}\right|_{(0,5)}$
$= \sqrt{5}/5$

$y - 5 = \frac{\sqrt{5}}{5}(x - 0)$

finding dy/dx

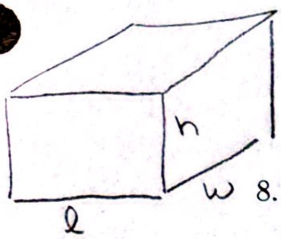
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-5}$

$\left.\frac{dy}{dx}\right|_{(0,5)}$ we need to find $t \ni 5 = t^2 \Rightarrow t = \pm\sqrt{5}$

note $(\pm\sqrt{5})^3 - 5(\pm\sqrt{5}) = 0$ ✓

$\left.\frac{dy}{dx}\right|_{(0,5)} = \left.\frac{dy}{dx}\right|_{t=\pm\sqrt{5}}$ we'll choose $+\sqrt{5}$

$\left.\frac{dy}{dx}\right|_{t=+\sqrt{5}} = \frac{2\sqrt{5}}{3\cdot 5 - 5} = \frac{2\sqrt{5}}{10} = \frac{\sqrt{5}}{5}$



8. Find the maximum and minimum volumes of a rectangular box with the constraints that the surface area is 1500cm^2 and total edge length is 200cm.

V = Maximize Volume $= l \cdot w \cdot h$

Constraints: $2lh + 2lw + 2hw = 1500$

$4l + 4w + 4h = 200$

Could do a series of substituting but I'll use Lagrange Multipliers

$\nabla(lwh) = \lambda \nabla(2lh + 2lw + 2hw) + \mu \nabla(4l + 4w + 4h)$

$$\left\{\begin{array}{l} wh = \lambda(2h+2w) + \mu(4) \\ lh = \lambda(2l+2h) + \mu(4) \\ lw = \lambda(2l+2h) + \mu(4) \\ 2lh + 2lw + 2hw = 1500 \\ 4l + 4w + 4h = 200 \end{array}\right\} \text{5 unknowns and 5 equations}$$

solve for μ with eq1: $\frac{1}{4}(wh) - \frac{1}{4}\lambda(2h+2w)$

sub into eq 2–5

$$\left.\begin{array}{l} lh = \lambda(2l+2h) + wh - \lambda(2h+2w) \\ lw = \lambda(2l+2h) + wh - \lambda(2h+2w) \\ 2lh + 2lw + 2hw = 1500 \\ 4l + 4w + 4h = 200 \end{array}\right\} \text{4 unknowns and 4 equd.}$$

Ug... not going to have enough space to solve this whole system.

$\leadsto$

To check we have a maximum I'd like to plug in a few points nearby to check

9. Common blood types are determined by three alleles, A, B, and O. If p is the percent of allele A in the population, q is the percent of allele b in the population and r is the percent of allele O in the population then the proportion of individuals with a mixed blood type (e.g. AB, AO or BO) is $P(p,q,r) = 2pq + 2pr + 2qr$. Find the maximal P value.

Maximize $P = 2pq + 2pr + 2qr$

Given that $p + q + r = 1$. (☺)

Since we used Lagrange above I'll use straight substitution here (But Lagrange would work?)

$p + q + r = 1 \Rightarrow p = 1 - q - r$

sub into P's equation

$P = 2(1-q-r)q + 2(1-q-r)r + 2qr$

$= 2q - 2q^2 - 2rq + 2r - \cancel{2qr} - 2r^2 + \cancel{2qr}$

$= -2q^2 + 2q - 2rq + 2r - 2r^2$

Finding the Critical Points

$$\left.\begin{array}{l} \frac{\partial P}{\partial q} = -4q + 2 - 2r = 0 \\ \frac{\partial P}{\partial r} = -2q + 2 - 4r = 0 \end{array}\right\} \text{2 unknowns and 2 equations}$$

eq 1 solve for r

$-2q + 1 = r$ (✱)

sub into eq2

$-2q + 2 - 4(-2q+1) = 0 \Rightarrow -2q + 2 + 8q - 4 = 0$

$\Rightarrow 6q - 2 = 0 \Rightarrow q = 2/6$ or $1/3$

Since (✱) $\Rightarrow r = -2q + 1$, when $q = 1/3$, $r = 1/3$

Since (☺) $\Rightarrow p + q + r = 1$ we'll have $p = 1/3$

Critical Point at $(1/3, 1/3, 1/3)$

Max? Note $P(1/3, 1/3, 1/3) = 2/3 \approx .67$

$P(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}) = .611$ $P(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}) = .611$ $P(1/3, 1/6, \frac{1}{2}) = .611$

$\Rightarrow$ likely max! (Could be saddle but unlikely w/ all checks)

10. Consider the double integral

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2 x}\, dx dy$$

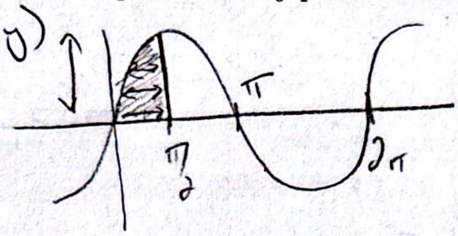
(a) Sketch the region in the xy-plane where the integral is taken over.

$x = \arcsin(y)$

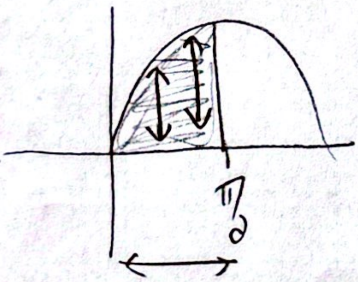
$\Rightarrow$ $y = \sin(x)$

and

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



(b) Switch the order of integration.



$$\int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos(x)\sqrt{1+\cos^2 x}\; dy\, dx$$

(c) Compute the double integral.

Integrating wrt y looks easier to me..

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin x} \cos(x)\sqrt{1+\cos^2 x}\; dy\, dx = \int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2 x}\; y\Big]_0^{\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2 x}\cdot \sin x - 0\, dx = \int_0^{\frac{\pi}{2}} \sin x \cos(x)\sqrt{1+\cos^2 x}\; dx$$

let $u = 1+\cos^2 x$

$du = -2\cos x \sin x\, dx$

$\Rightarrow -\frac{1}{2} du = \cos x \sin x\, dx$

So $\int \sin x \cos(x) \sqrt{1+\cos^2 x}\, dx = \int \sqrt{u}\left(-\frac{1}{2}\right) du = -\frac{1}{2}\int u^{\frac{1}{2}} du = -\frac{1}{2} u^{\frac{3}{2}} \cdot \frac{2}{3} + C$

$= -\frac{1}{3}\left(1+\cos^2 x\right)^{3/2} + C$

So back to original:

$$\int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\sqrt{1+\cos^2 x}\, dx = -\frac{1}{3}\left(1+\cos^2 x\right)^{\frac{3}{2}}\Big]_0^{\frac{\pi}{2}} = -\frac{1}{3}\left(1+\cos^2\left(\frac{\pi}{2}\right)\right)^{\frac{3}{2}} + \frac{1}{3}\left(1+\cos^2 0\right)^{\frac{3}{2}} = -\frac{1}{3} + \frac{1}{3}\cdot 2^{\frac{3}{2}} = \frac{-1+\sqrt{8}}{3}$$