Fix 4c (remove)

Key

Exam 2

TMath 126

Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) Let f be a function of x and y. If $\nabla f(c,d) = (2,1)$, then the vector $\langle 2,1\rangle$ is tangent to the contour line of the surface of f at $(c, d, f(c,d))$.

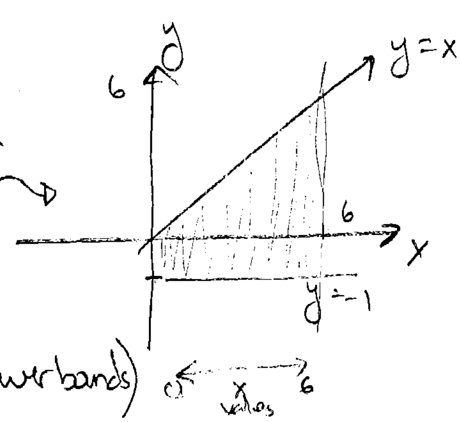
   False $\nabla f(c,d)$ points in the direction of steepest assent. The contour line would keep the "elevation"/z value constant. So $\nabla f(c,d)$ is $\perp$ to contour line

   (b) $\int_{-1}^{2}\int_{0}^{6} x^2\sin(x-y)\,dx\,dy = \int_{0}^{6}\int_{-1}^{2} x^2\sin(x-y)\,dy\,dx$

   True. The function $x^2\sin(x-y)$ is continuous on the rectangle $[0,6]\times[-1,2]$ so we can use Fubini's Thm.
   (or we could compute each individually? ... would involve integration by parts on one side...)

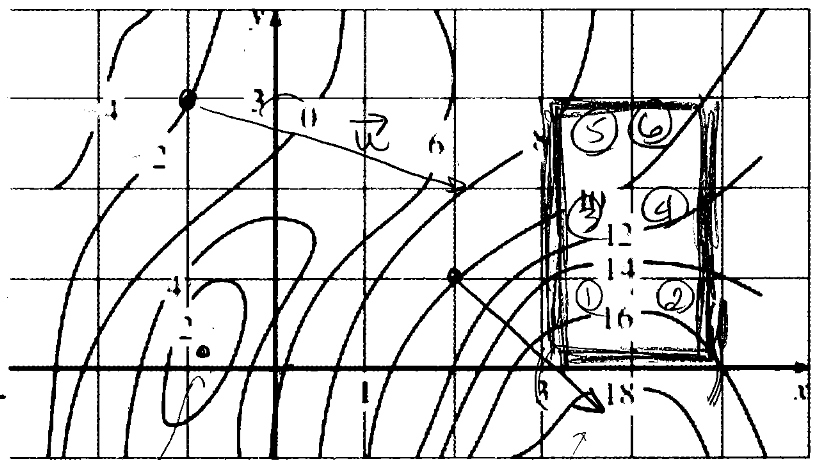
   (c) $\int_{-1}^{x}\int_{0}^{6} x^2\sin(x-y)\,dx\,dy = \int_{0}^{6}\int_{-1}^{x} x^2\sin(x-y)\,dy\,dx$

   False. The integral corresponds with the volume above the area
   If we reversed the order of integration we'll have to split the computation (b/c 2 dif. lowerbounds)

   $$\int_0^6\int_y^6 x^2\sin(x-y)\,dx\,dy + \int_{-1}^{0}\int_0^6 x^2\sin(x-y)\,dy\,dx$$

2. Let f have the contour lines shown on the right.



(a) Estimate $f(2,1)$

10

(b) Sketch the direction of the vector $\nabla f(2,1)$ on the graph.

length does not matter here

$\perp$ to contour line

points in direction of steepest increase

probably should be longer if length was asked

(c) Identify one critical point on the graph of f and identify it as a local minimum, maximum or neither.

$\approx (-.8, .2)$ looks like local minimum

(d) Let $\vec{u} = \langle 3, -1 \rangle$ Determine whether the directional derivative of f at point $(-1,3)$ along $\vec{u}$ is positive, negative, or zero.
Justify your answer.

positive

at $(-1,3)$ the z-coord is -2.

As we move along $\vec{u}$ the z values are increasing, so up?

positive change in z

(e) Estimate the volume bounded by f above the rectangle $3 \leq x \leq 5$ and $0 \leq y \leq 3$. Be clear about what choices you are making to estimate the volume.

region is outlined

I'll break this up into 6 regions where $\Delta x = 1$ and $\Delta y = 1$

To determine the heights I'll estimate the heights in the center of each square

$9 \cdot 1 \cdot 1 + 10 \cdot 1 \cdot 1 + 12 \cdot 1 \cdot 1 + 13 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 = 76$

so ≈ 76 units^3

3. You are given the following data of a function $g(x, y)$. Your boss wants you to approximate $g(.8, 1.4)$ and wants to be convinced you're doing something sophisticated. Find a linear approximation for your boss and explain your choices (there are many that you will make!). We use 3D linear approximation

$$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$$

x	y	$g(x,y)$
0.55	1.2	27
0.65	1.0	31
0.65	1.1	29
0.75	1.2	50

use this for x_0, y_0, z_0 b/c closest to $x=.8$ & $y=1.4$

$$m_x = \frac{dz}{dx}\Big|_{y=1.2} \approx \frac{\Delta z}{\Delta x} = \frac{50-27}{.75-.55} = \frac{23}{.2} = 115$$

need to keep y const, so use

$$m_y = \frac{dz}{dy}\Big|_{x=.65} \approx \frac{31-29}{1.0-1.1} = \frac{2}{-.1} = -20$$ to keep x const use 1st & 3rd row

So $z - 50 = 115(x - .75) - 20(y - 1.2)$ now plug in $.8 = x$ & $1.4 = y$ to get approx

4. Consider the double integral

$$\int_0^1 \int_{\arcsin y}^{\frac{\pi}{2}} \cos(x)\sqrt{1 + \cos^2 x}\, dx dy$$

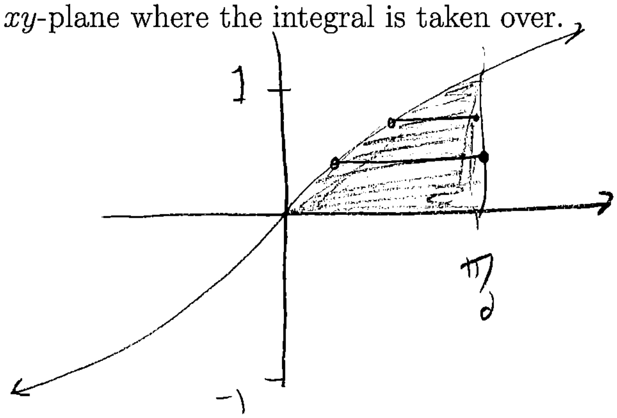
(a) Sketch the region in the xy-plane where the integral is taken over.

$x = \arcsin(y)$

$\Leftrightarrow$

$\sin(x) = y$

where $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

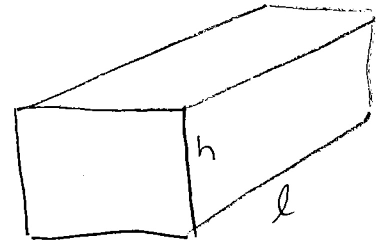


(b) Switch the order of integration.

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin(x)} \cos(x)\sqrt{1 + \cos^2 x}\, dy\, dx$$

(c) Compute the double integral. With technology right?

3. Find the maximum and minimum volumes of a rectangular box with the constraints that the surface area is 1500cm^2 and total edge length is 200cm.



$V =$ (Maximize) volume $= l \cdot w \cdot h$

Constraints: $2lh + 2lw + 2hw = 1500$

$4l + 4w + 4h = 200$

could do a series of substituting but I'll use Lagrange Multipliers

$\nabla V = \nabla(lwh) = \lambda \nabla (2lh + 2lw + 2hw) + \mu \nabla (4l + 4w + 4h)$

$$\begin{cases} wh = \lambda(2h + 2w) + \mu(4) & \text{(wrt } l\text{)} \\ lh = \lambda(2l + 2h) + \mu(4) & \text{(wrt } w\text{)} \\ lw = \lambda(2l + 2w) + \mu 4 & \text{(wrt } h\text{)} \\ 2lh + 2lw + 2hw = 1500 \\ 4l + 4w + 4h = 200 \end{cases}$$

5 equations / 5 unknowns

4. Common blood types are determined by three alleles, A, B, and O. If p is the percent of allele A in the population, q is the percent of allele B in the population and r is the percent of allele O in the population then the proportion of individuals with a mixed blood type (e.g. AB, AO or BO) is $P(p, q, r) = 2pq + 2pr + 2qr$. Find the maximal P value.

Maximize $P = 2pq + 2pr + 2qr$

note $p + q + r = 1$ ← total population

(% of A, % of B, % of O)

More Lagrange? Maybe I'll use substitution instead.

$p + q + r = 1 \Rightarrow p = 1 - q - r$

$\Rightarrow P = 2(1-q-r)q + 2(1-q-r)r + 2qr$

$= 2q - 2q^2 - 2rq + 2r - \cancel{2qr} - 2r^2 + \cancel{2qr}$

$= -2q^2 + 2q - 2rq + 2r - 2r^2$

To optimize function of 2 variables

$\frac{\partial P}{\partial q} = 0 \Rightarrow -4q + 2 - 2r = 0$

AND

$\frac{\partial P}{\partial r} = 0 \Rightarrow -2q + 2 - 4r = 0$

Finding Critical Points $\begin{cases} -4q + 2 - 2r = 0 & (1) \\ -2q + 2 - 4r = 0 & (2) \end{cases}$

Use (1) to solve for $r \Rightarrow \frac{-4q + 2}{2} = r$

$r = -2q + 1$

Sub into (2) $-2q + 2 - 4(-2q + 1) = 0$

$-2q + 2 + 8q - 4 = 0$

$6q = 2 \Rightarrow q = \frac{1}{3}$

Since $r = -2(q) + 1 = -2(\frac{1}{3}) + 1 = -\frac{2}{3} + 1 = \frac{1}{3}$

Since $p = 1 - \frac{1}{3} - \frac{1}{3} \Rightarrow p = \frac{1}{3}$

Critical Point: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ Max value?

Use 2nd derivative Test

$\frac{\partial P^2}{\partial^2 q} = -4 \quad \frac{\partial P^2}{\partial^2 r} = -4 \quad \frac{\partial P^2}{\partial q \partial r} = -2$

$D = (-4)(-4) - (-2)^2 = 16 - 4 > 0$

$\frac{\partial P^2}{\partial^2 q} = -4 < 0 \Rightarrow$ max value @

$P(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 2\cdot\frac{1}{9} + 2\frac{1}{9} + 2\frac{1}{9} = \frac{6}{9} = \boxed{\frac{2}{3}}$