Key

Exam 2 **TMath 126** Practice Spr 2025

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

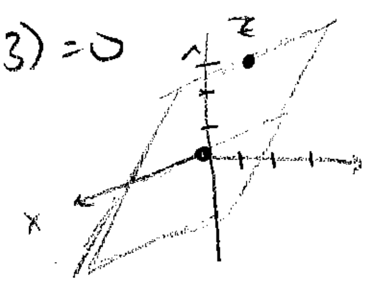
   (a) Let $f$ be a function of $x$ and $y$. If $\nabla f(c,d) = (2,1)$, then the vector $\langle 2,1\rangle$ is tangent to the contour line of the surface of $f$ at $(c,d,f(c,d))$.

   False? $\nabla f(c,d)$ points in the direction of steepest assent.
   The contour lines would keep the "elevation"/z-value constant.
   So $\nabla f(c,d)$ would be $\perp$ to the contour line.

   (b) If $f_x(2,3) = 0$ then $(2,3,f(2,3))$ will be a critical point.

   False, Both $f_x(2,3)=0$ AND $f_y(2,3)=0$

   eg $f(x,y)=3y$ note $f_x(2,3)=0$
   but $f_y(2,3)=3$




   (c) To optimize the function $f(x,y) = e^{xy}$ subject to the constraint $x^3+y^3=16$ we would need to solve the following system of equations:

   $C(x,y) = x^3+y^3$

   $f_x(x,y) = e^{xy}\frac{d}{dx}(xy)$
   $= y e^{xy}$
   (chain)

$$\left\{\begin{array}{c} ye^{xy} = 3x^2 \\ xe^{xy} = 3y^2 \\ x^3+y^3 = 16 \end{array}\right\} \qquad (1)$$

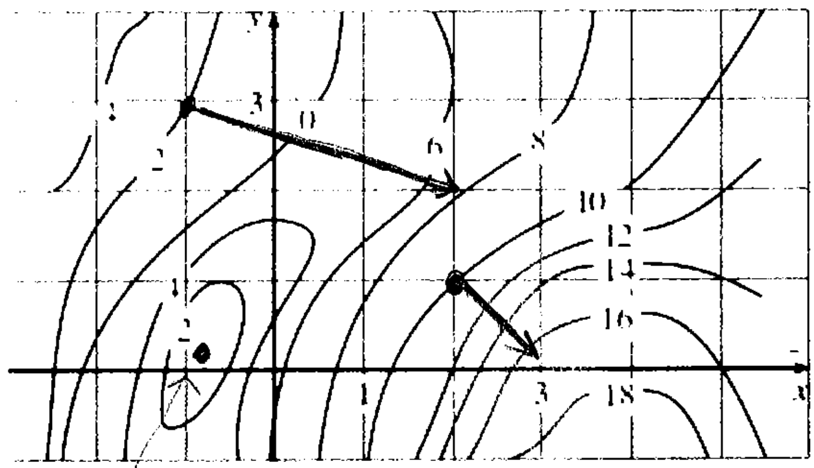
   missing the $\lambda$ scalars

   We can use Lagrange multipliers

$$\left\{\begin{array}{l} f_x(x,y) = \lambda C_x(x,y) \\ f_y(x,y) = \lambda C_y(x,y) \\ C(x,y) = 16 \end{array}\right\} = \left\{\begin{array}{l} ye^{xy} = \lambda 3x^2 \\ xe^{xy} = \lambda 3y^2 \\ x^3+y^3 = 16 \end{array}\right\}$$

   False.

2. Let $f$ have the contour lines shown on the right.




(a) Estimate $f(2, 1)$

$\approx 10$

(b) Sketch the direction of the vector $\nabla f(2, 1)$ on the graph.

→ length is not a worry

$\perp$ to contour lines

points in direction of steepest increase in $z$

(c) Identify one critical point on the graph of $f$ and identify it as a local minimum, maximum or neither.

$\approx (-.8, .2)$ looks like local minimum

(d) Let $\vec{u} = \langle 3, -1 \rangle$ Determine whether the directional derivative of $f$ at point $(-1, 3)$ along $\vec{u}$ is positive, negative, or zero. *Justify* your answer.

$D_{\vec{u}}(-1,3)$

Positive

@ $(-1,3)$ the $z$-coord is $-2$.

When we travel in direction of $\langle 3,-1 \rangle$ the $z$ values go up?

so positive change in $z$.

(e) Estimate a point $(a, b)$ so that $f_x(a, b) > 0$ and $f_y(a, b) > 0$.

The local minimum would work $\approx(-.8, .2)$

since as we move in positive $y$ direction, $z$-values ↑

as we move in positive $x$-direction, $z$-values ↑

3. You are given the following data of a function $g(x, y)$. Your boss wants you to approximate $g(.8, 1.4)$ and wants to be convinced you're doing something sophisticated. Find a linear approximation for your boss and explain your choices (there are many that you will make!). We use 3D linear approximation

| $x$ | $y$ | $g(x,y)$ |
|---|---|---|
| 0.55 | 1.2 | 27 |
| 0.65 | 1.0 | 31 |
| 0.65 | 1.1 | 29 |
| 0.75 | 1.2 | 50 |

← use this for $x_0, y_0, z_0$ b/c closest to $x=.8$ and $y=1.4$

$$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$$

$$m_x = \left.\frac{dz}{dx}\right|_{y=1.2} \approx \frac{50-27}{.75-.55} = \frac{23}{.2} = 115$$

↳ closest can get to $y = 1.4$ & fix $y$ to one value

$$m_y = \left.\frac{dz}{dy}\right|_{x=.65} \approx \frac{31-29}{1.0-1.1} = \frac{2}{-.1} = -20$$

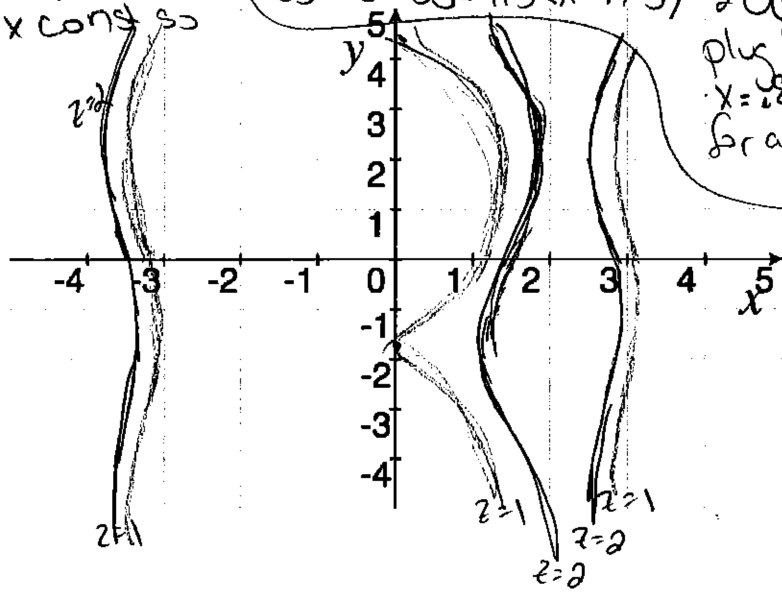
need to keep $x$ const so

So $z - 50 = 115(x - .75) - 20(y - 1.2)$

plug in $x = .8$ & $y = 1.4$ for approx

4. Let $f(x, y) = x^2 \sin(x) - \sin(y)$.

(a) Draw sections of the contour map/the level curves of $f$ when $z = 1$ and $z = 2$. Label the curves!




(b) [3] Find $f_x(x, y)$

$$f_x(x,y) = \underbrace{x^2\cos(x) + 2x\sin(x)}_{\text{product rule}} + \underbrace{0}_{y \text{ is a constant}}$$

(c) Find $D_{\vec{u}}(2, 0)$ where $\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$. note $\|\vec{u}\| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$ ✓

So $D_{\vec{u}}(2,0) = \nabla f(2,0) \cdot \vec{u}$

$$= \left.\langle x^2\cos(x) + 2x\sin(x), -\cos(y) \rangle\right|_{(2,0)} \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

$$= \langle 4\cos(2) + 4\sin(2), -1 \rangle \cdot \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle = 2\sqrt{3}(\cos(2) + \sin(2)) + \frac{-1}{2}$$

For the following problem you will outline (not actually find!) a solution. Make sure your outline includes:

(a) definitions of variables used,

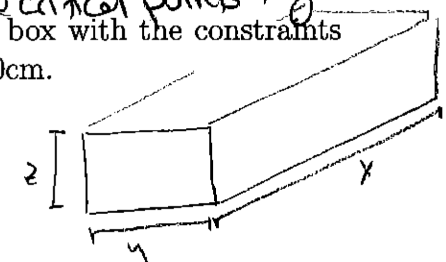
(b) identifying the function that needs to be optimized,

(c) boxing systems of equations that need to be solved (but do not solve them!), &

(d) explaining how you would verify your work is correct (ie a maximum)

↳ I'd plug in a bunch of values around the critical points to get a feel of V.

5. Find the maximum and minimum volumes of a rectangular box with the constraints that the surface area is 1500cm$^2$ and total edge length is 200cm.

a) 


b) Optimize Volume $V = x \cdot y \cdot z$

c) Constraint Surface area = 1500cm² $\Rightarrow 2zy + 2xy + 2zx = 1500\text{cm}^2$

Edge length = 200cm $\Rightarrow 4x + 4y + 4z = 200\text{cm}$

could do a series of substitutions but I'll use Lagrange Multiples

$\nabla V = \nabla \text{constraints} = \lambda \nabla(2zy + 2xy + 2zx) + \mu \nabla(4x + 4y + 4z)$

we'll have 5 equations and 5 unknowns

$$\boxed{\begin{array}{ll} yz = \lambda(2y + 2z) + \mu 4 & 2zy + 2xy + 2zx = 1500 \\ xz = \lambda(2z + 2x) + \mu 4 & 4x + 4y + 4z = 200 \\ xy = \lambda(2y + 2x) + \mu 4 & \end{array}}$$

6. Common blood types are determined by three alleles, $A$, $B$, and $O$. If $p$ is the percent of allele $A$ in the population, $q$ is the percent of allele $B$ in the population and $r$ is the percent of allele $O$ in the population then the proportion of individuals with a mixed blood type (e.g. $AB$, $AO$ or $BO$) is $P(p, q, r) = 2pq + 2pr + 2qr$. Find the maximal $P$ value. a)

b) Maximize $P(p,q,r) = 2pq + 2pr + 2qr$

note the constraint: $p + q + r = 1$ → total population — could be used to reduce # of variables

(% of A, % of B, % of O)

c) I could use Lagrange but I did that above so let's do a dif approach

$$\left\{\begin{array}{l} P_p(p,q,r) = 0 \\ P_q(p,q,r) = 0 \\ P_r(p,q,r) = 0 \end{array}\right\} \quad \boxed{\begin{array}{l} 2q + 2r = 0 \\ 2p + 2r = 0 \\ 2p + 2q = 0 \end{array}}$$

d) 2nd derivative test wouldn't work with 3 variables so I'd have to plug in a few (p,q,r) around the critical points to verify we have a max.