Key

Exam 1 | TMath 126 | Practice

Note: This is a practice exam and is intended only for study purposes. The actual exam will contain different questions and may have a different layout.

1. TRUE/FALSE: Identify a statement as True in each of the following cases if the statement is *always* true and provide a brief justification. Otherwise, identify it as false and provide a counterexample.

   Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$.
   Recall that $\cdot$ refers to the dot product, and $\times$ refers to the cross product.

   (a) If $\vec{u} \cdot \vec{v} = 0$, then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

   false let $\vec{u} = \langle 1,0,0 \rangle$ and $\vec{v} = \langle 0,1,0 \rangle$.
   Then $\vec{u} \cdot \vec{v} = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$ but neither $\vec{u}$ or $\vec{v}$ is the zero vector.

   (b) $(\vec{u} \times \vec{w}) \cdot \vec{w} = 0$

   True $\vec{u} \times \vec{w}$ produces a vector $\perp$ to both $\vec{u}$ and $\vec{w}$. Since $(\vec{u} \times \vec{w})$ is $\perp$ to $\vec{w}$, the dot product of the two vectors is : $(\vec{u} \times \vec{w}) \cdot \vec{w} = \|\vec{u} \times \vec{w}\| \cdot \|\vec{w}\| \cdot \cos(90°) = 0$.

   (c) $\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{\vec{u}}{\|\vec{u}\|} \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$

   True.
   $$\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1}{\|\vec{u}\|\|\vec{v}\|}(u_1v_1 + u_2v_2 + u_3v_3) = \left(\frac{u_1}{\|\vec{u}\|}\right)\left(\frac{v_1}{\|\vec{v}\|}\right) + \left(\frac{u_2}{\|\vec{u}\|}\right)\left(\frac{v_2}{\|\vec{v}\|}\right) + \left(\frac{u_3}{\|\vec{u}\|}\right)\left(\frac{v_3}{\|\vec{v}\|}\right) = \frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

   (d) The line $(2 + 3t, -4t, 5 + t)$ where $t \in \mathbb{R}$ intersects the plane $4x + 5y - 2z = 18$ at the point $(-4, 8, 3)$. Want point that satisfies both conditions? so $x = 2 + 3t$ and satisfies $4x + 5y - 2z = 18$. Similarly for y & z.
   Substitution $\Rightarrow 4(2+3t) + 5(-4t) - 2(5+t) = 18$
   $8 + 12t - 20t - 10 - t = 18 \Rightarrow -9t - 2 = 18 \Rightarrow -9t = 20 \Rightarrow t = \frac{20}{9}$
   when $t = 20/9$ the line has coord $\left(2+3\left(\frac{20}{9}\right), -4\left(\frac{20}{9}\right), 5 + \frac{20}{9}\right) \neq (-4, 8, 3)$
   So no? False. Note: $(-4,8,3)$ is not on the line I think..

   (e) If $\vec{r}(t) = \langle t^2, \ln(et), t^3 - 3t \rangle$, then the line tangent to $\vec{r}(1)$ is:

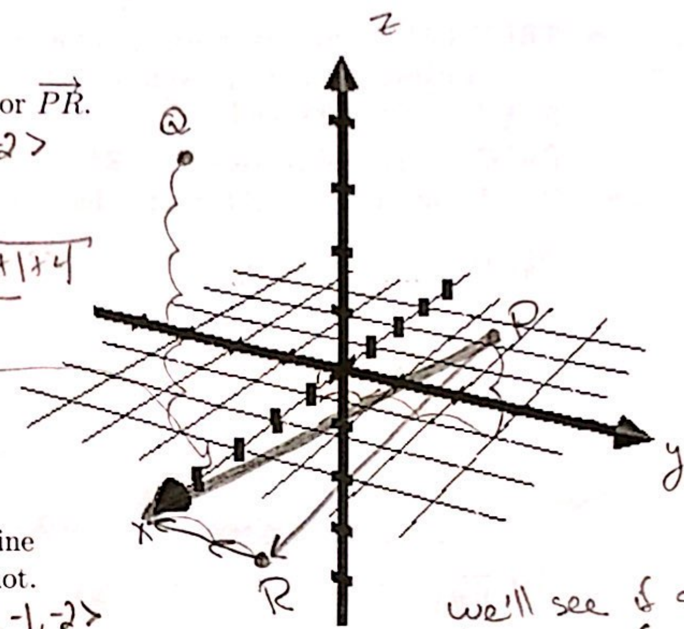
   $$\langle x, y, z \rangle = \langle 1, 1, -2 \rangle + \langle 2t, \frac{e}{t}, 3t^2 - 3 \rangle$$ ← not a line?

   False. Note $\vec{r}'(t)$ is $\langle 2t, \frac{1}{et} \cdot e, 3t^2 - 3 \rangle = \langle 2t, \frac{1}{t}, 3t^2 - 3 \rangle$
   A line has a directional vector comprised of numbers so directional vector would be $\langle 2(1), \frac{1}{1}, 3(1) - 3 \rangle = \langle 2, 1, 0 \rangle$
   ($r(1)$) ($r'(1)$)
   So line would be $\langle x, y, z \rangle = \langle 1, 1, -2 \rangle + s\langle 2, 1, 0 \rangle$

Show your work for the following problems. The correct answer with no supporting work will receive NO credit.

2. Consider the points: $P(1,3,2)$, $Q(3,-1,6)$, and $R(5,2,0)$. Also let $S(3,6,1.5)$ and $T(-9,-14,-12.5)$.

   (a) Plot the points P, Q, and R.



   (b) Find the components of the vector $\overrightarrow{PR}$.

   $\langle 5-1, 2-3, 0-2\rangle = \langle 4,-1,-2\rangle$

   (c) Find the length of $\overrightarrow{PR}$.

   $\sqrt{(5-1)^2+(2-3)^2+(0-2)^2} = \sqrt{16+1+4}$
   $= \sqrt{21}$

   (d) Draw the vector $\overrightarrow{PR} - 2\vec{j}$ and then write its components.

   $\langle 4,-1,-2\rangle - 2\langle 0,1,0\rangle$

   $\langle 4,-3,-2\rangle$

   (e) Use calculus methods to determine if $\triangle PQR$ is a right triangle or not.

   non calculus method
   find $|PQ|$, $|PR|$ and $|QR|$.
   See if the lengths of edges satisfy $a^2+b^2=c^2$.
   Recall length of $\triangle$
   $a^2+b^2=c^2 \iff$ right $\triangle$

   $\overrightarrow{PR} = \langle 4,-1,-2\rangle$
   $\overrightarrow{PQ} = \langle 3-1, -1-3, 6-2\rangle = \langle 2,-4,4\rangle$
   $\overrightarrow{QR} = \langle 5-3, 2-1, 0-6\rangle = \langle 2,3,-6\rangle$

   we'll see if any of the pairs form a 90° with the dot product.

   $\overrightarrow{PR}\cdot\overrightarrow{PQ} = 4\cdot 2+4-8 \neq 0$
   $\overrightarrow{PR}\cdot\overrightarrow{QR} = 4\cdot 2-3+12 \neq 0$
   $\overrightarrow{PQ}\cdot\overrightarrow{QR} = 4-12-24 \neq 0$

   no 90° angles (happen when dot prod $=0$) so not a right $\triangle$.

   (f) Find the equation of the plane that passes through P, R, and Q.

   Looking for $\eta\cdot(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle) = 0$
   need η to be $\perp$ to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$
   the cross product will give $\perp$ vector

   $$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = \vec{i}(8+4) - \vec{j}(-4-16) + \vec{k}(-2+16)$$
   $$= 12\vec{i} + 20\vec{j} + 14\vec{k}$$

   so $\langle 12,20,14\rangle\cdot(\langle x,y,z\rangle - \langle 3,-1,6\rangle) = 0$ works

   note: there are lots of answers here!

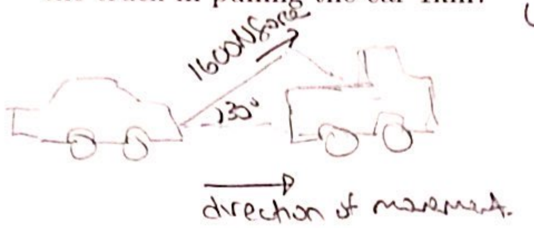
   (g) Does the line that passes through S and T intersect the plane you found in part (a)? Justify yourself.

   $\overrightarrow{TS} = \langle 3+9, 6+14, 14\rangle$
   $= \langle 12, 20, 14\rangle$

   looks // to η found above!

   so yeah, the line $\overleftrightarrow{ST}$ will intersect the plane in (f)

3. A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and tension in the chain is 1600 Newtons. How much work (in J) is done by the truck in pulling the car 1km?



$\text{Work} = \vec{F} \bullet \vec{d}$

$= \|\vec{F}\| \cdot \|\vec{d}\| \cos\theta$

$= 1600\text{N} \cdot 1\text{km} \cdot \cos 30°$

$= 1600 \frac{\sqrt{3}}{2} = 800\sqrt{3}\ \text{J}$

OR $\vec{F} = \langle 1600\cos 30°, 1600\sin 3$

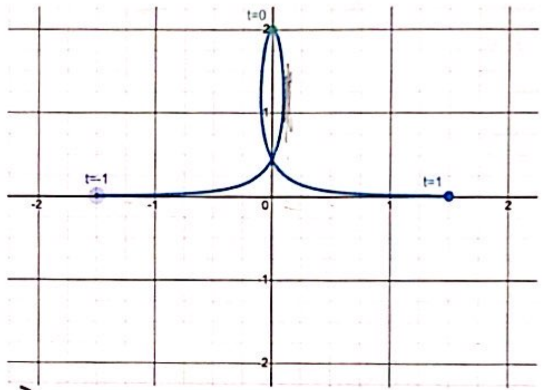
Sohcahtoa

$\cos 30 = \frac{F_x}{1600}$

$\therefore \vec{d} = \langle 1, 0 \rangle$

$\vec{F} \cdot \vec{d} = 1600\cos 30 = 800\sqrt{3}$

4. Consider the parametric curve $x = f(t)$, $y = g(t)$ where $-1 \le t \le 1$, graphed below for the following questions.



(a) Looking at the graph, approximate where $\frac{dy}{dx}$ is not defined. (Report either a point on the graph or an approximate t value.)

endpoints $t = 1$ & $t = -1$

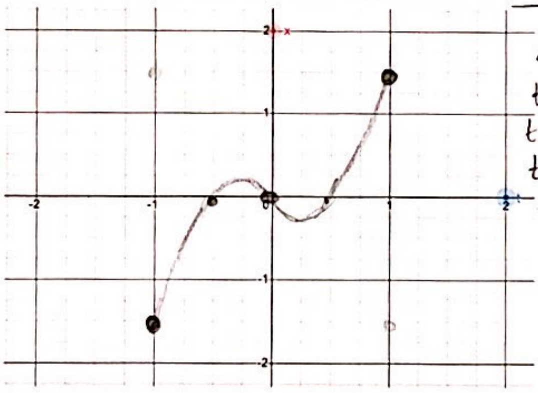
$\approx (1.5, 0)$ $\approx (-1.5, 0)$

and vertical tangent lines

@ $t = 1/4$ and $-1/4$

$\approx (.1, 1.2)$ $\approx (-.1, 1.2)$

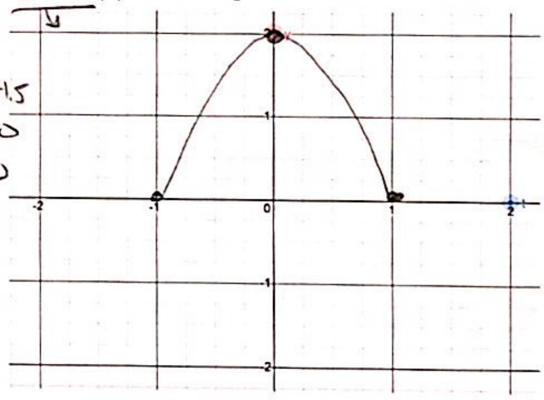
(b) Sketch the equations $x = f(t)$ and $y = g(t)$ on the pair of axis below.



$t = -1 \Rightarrow x = -1.5$

$t = -.5 \Rightarrow x \approx 0$

$t = 0 \Rightarrow x = 0$



y values always pos

(c) Given the following information, find the (approximate) line tangent to the curve $x = f(t)$, $y = g(t)$ when $t = \frac{1}{2}$. Use whatever form of a line you like (eg. parametric, slope-intercept, standard, etc)

$f\left(\frac{1}{2}\right) \approx 0$ $\quad g\left(\frac{1}{2}\right) \approx .45$ $\quad f'\left(\frac{1}{2}\right) \approx 1$ $\quad g'\left(\frac{1}{2}\right) \approx -2.68$

note f' is derivative with respect to t

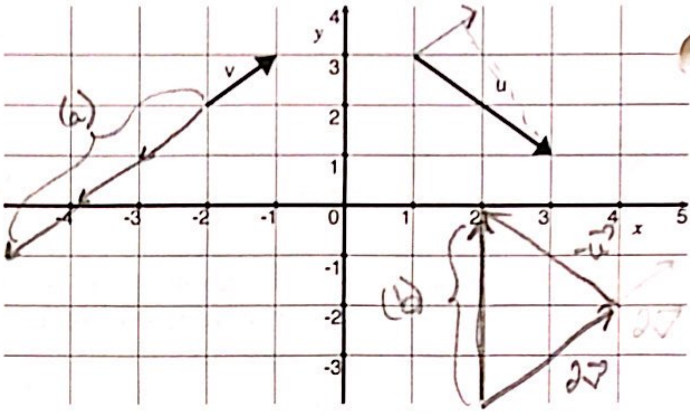
There are Lots of correct answers here!

$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \frac{\left.\frac{dy}{dt}\right|_{t=\frac{1}{2}}}{\left.\frac{dx}{dt}\right|_{t=\frac{1}{2}}} = \frac{g'(\frac{1}{2})}{f'(\frac{1}{2})} = \frac{-2.68}{1} = -2.68$

$y - y_1 = m(x - x_1)$

$y - .45 = -2.68(x - 0)$

4. Consider the vector $\vec{v}$ and $\vec{u}$ shown to the right.



(a) Draw the vector $-3\vec{v}$.

(b) Draw the vector $2\vec{v} - \vec{u}$.

(c) Find the projection of $\vec{u}$ onto $\vec{v}$.

... are these $\perp$?

$\langle 1,1\rangle \circ \langle 2,-2\rangle$

$= 2-2 = 0$

perpendicular ?! :)

projection will be $\vec{0}$

OR $\quad \text{Proj}_{\vec{v}}\, \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\, \vec{v}$

$= \dfrac{0}{\sqrt{2}^2} \langle 1,1\rangle = \vec{0}$

5. We define $\vec{r}(t)$ by: $x(t) = 1 + t^4$, $y(t) = te^{-t}$, and $z(t) = \sin(2t)$.

(a) Find the line tangent to the curve $\vec{r}(t)$ when $t = 0$.

looking for a line $\langle x_0, y_0, z_0\rangle + s\vec{d}$ where $s \in \mathbb{R}$

when $t=0$ $\quad \vec{r}(0) = \langle 1,0,0\rangle$ so $\quad \langle 1,0,0\rangle + s\vec{d}$ where $s \in \mathbb{R}$

need directional vector so $\quad (x'(0), y'(0), z'(0)) = \langle 0,1,2\rangle$

$x'(t) = 0 + 4t^3 \Rightarrow x'(0) = 0$

$y'(t) = t(-1)e^{-t} + e^{-t} \Rightarrow y'(0) = 0+1$

$z'(t) = 2\cos(2t) \Rightarrow z'(0) = 2$

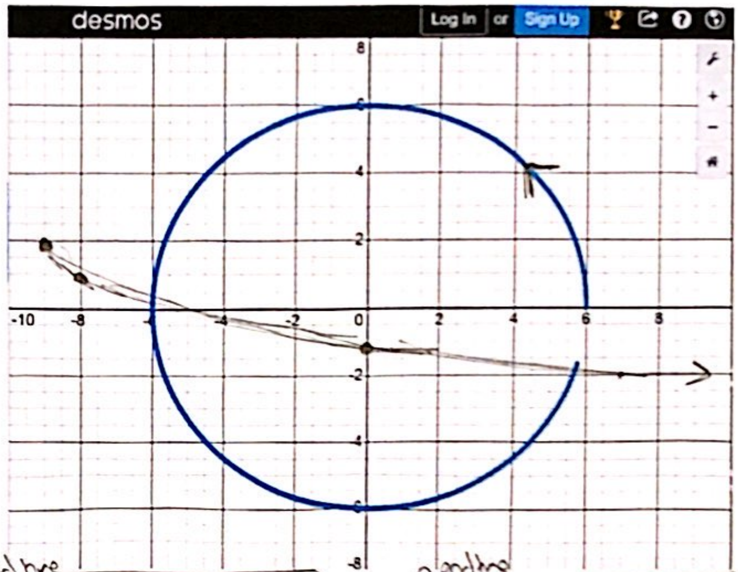
so $\langle 1,0,0\rangle + s\langle 0,1,2\rangle$

(b) Find the length of the arc traced by $\vec{r}(t)$ from $t = 0$ to $t = 5$.

$$\int_0^5 \sqrt{(4t^3)^2 + (e^{-t} - te^{-t})^2 + (2\cos(2t))^2}\, dt = 626$$

(technology)

7. A plane's position is traced by a parameterized curve: $x_p(t) = t^2 - 9$ and $y_p(t) = 2 - t$ (in km). Similarly, parameterized curves for a helicopter's position is $x_h(t) = 6\cos(t)$ and $y_h(t) = 6\sin(t)$ (in km). The helicopter's path is traced below for $t = 0$ to 10.



(a) As t increases, indicate the direction of the helicopter's path by adding an arrow to the path graphed.

(b) Sketch the path of the plane from $t = 0$ to $t = 8$.

$t=0 \Rightarrow (0-9, 2-0)$ $\quad t=3 \Rightarrow (0,-1)$

$t=1 \Rightarrow (-8, 1)$ $\quad t=10 \Rightarrow (91,$

(c) Set up the expression that will return the distance traveled by the helicopter between (6, 0) and (5.6568, −2). Make sure your answer can be completed with technology, you do *not* need to find the numeric answer!

$\int_{\text{start time}}^{\text{end time}} \sqrt{(x_h'(t))^2 + (y_h'(t))^2}\, dt = \int_{\text{start time}}^{\text{end time}} \sqrt{(-6\sin t)^2 + (6\cos t)^2}\, dt$

$\frac{d}{dt}(6\cos t)$ $\quad \frac{d}{dt}(6\sin t)$

need to find start + end times

@ $t=0$, $x(0) = 6\cos(0) = 6$
$y(0) = 6\sin(0) = 0$
so $(6, 0)$

$\therefore$ $t=0$ is start point.

end point, find t so that
$(6\cos t, 6\sin t) = (5.6568, -2)$

xcoords $\quad$ ycoords

$6\cos t = 5.6568$ $\quad$ $6\sin t = -2$

$\Rightarrow t = .34$ or 5.943 $\quad$ $\Rightarrow t = 3.481$ or 5.94

So $\int_0^{5.943} \sqrt{(-6\sin t)^2 + (6\cos t)^2}\, dt \approx 35.658$ km (Desmos)

(d) Find the coordinates of any points where the two paths intersect.

To use Desmos I need rectangular coordinates?

plane: $t = 2 - y$ $\quad$ helicopter: $x^2 + y^2 = 36$

$\Rightarrow x = (2-y)^2 - 9$

(Desmos) $\Rightarrow$ $(5.712, -1.836)$ and $(-5.994, .266)$

(e) Does the plane ever collide with the helicopter? Provide justification for your answer.

We need to find t values for the plane and helicopter for the 2 points

helecopter:
$(5.712, -1.836) = (6\cos t, 6\sin t)$

$\Rightarrow t = ~~.34~~$ s or 5.972 s

plane:
$(5.712, -1.836) = (t^2 - 9, 2 - t)$

$\Downarrow$

$t = 3.836$ s

different times so do not hit although 2 seconds apart is really close...

Similar check for the point $(-5.994, .266)$